A Systematic Approach to Invariance in Twelve-Tone Music

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Integrating theories of twentieth-century music into the traditional four-semester undergraduate theory core presents several challenges to the instructor; at many institutions, however, the basic theory sequence provides the only vehicle for exploring the structure of more recent music. The problems include accelerating the pace of traditional harmony study and grappling with the immensity and diversity of the twentieth-century repertoire. While both problems are perplexing, the latter presents a more serious stumbling block for both instructors and students.

In a recent article David Mancini defines one danger encountered when incorporating twentieth-century theory in the basic sequence: after rigorous study of tonal harmony, a short survey of newer music seems superficial because of a "lack of precise theoretical and analytical tools traditionally available to students at this level."[1] Mancini focuses his discussion on the identification and comparison of pitch collections. Noting that labels such as *polychord* or *mixed-interval chord* have limited analytical application, he proposes a distillation of set theory in order to provide students with "a systematic method of presenting analytical conclusions."[2]

Mancini addresses two fundamental pedagogical issues: (1) What do we want students to learn about this music? and (2) What tools will

[1]David Mancini, "Teaching Set Theory in the Undergraduate Core Curriculum," *Journal of Music Theory Pedagogy* 5/1 (Spring 1991): 95.

[2]Ibid. In a related vein Joel Phillips suggests in his review of Stefan Kostka's *Materials and Techniques of Twentieth-Century Music* ["Review of Three Twentieth-Century Texts," *Journal of Music Theory Pedagogy* 5/2 (Fall 1991): 201-213] that Kostka spends an inordinate amount of time placing descriptive labels (mystic chord, quartal chord, etc.) on sonorities that can be described more precisely through set theory. Kostka does introduce set theory in his text; Phillips chides him for clinging to ambiguous descriptive labels as well. The conflict arises from the interaction of the traditional survey-oriented, descriptive approach to teaching twentieth-century music with the current trend toward utilizing more precise and meaningful analytical tools. Kostka attempts to cover both bases.

contribute most effectively to the learning process? Do we want the class to acquire a set of descriptive labels, or would we rather they have a systematic analytical method capable of elucidating musical relationships and structure? The answer seems clear, yet the time-constraint obstacle looms large. We cannot expect sophomores to deal with professional theoretical and analytical systems without risking general insurrection; but, as Mancini argues, we can with effort distill these theories into more accessible form.

My purpose in the present paper is to examine these two basic issues, the goals and the tools of learning, in light of the pedagogy of twelve-tone theory. Much of this pedagogy falls into the "descriptive labeling" category. Students learn what a row is, learn its basic transformations (transposition, inversion, retrograde, and retrograde inversion), learn how to construct a matrix of the forty-eight common row transformations, and learn to "chase" the row through a given composition. When they have completed their analysis, what have they learned about the music? They have learned about as much as they would by labeling every vertical sonority in the second movement of William Schuman's *Three-Score Set* as a polychord. Or, to make the point more dramatically, they have learned what they would by labeling tertian chords according to their scale-degree roots, yet not considering voice leading or functional relationships among the scale steps.

Little wonder that students often respond by branding twelve-tone music as meaningless mathematical machination. They have not been provided with tools that allow them to appreciate the musical side of serial music, but rather have been taught to do elementary math. Due often to time pressures, the student likely learns little of factors governing the composer's construction and utilization of the row, and of concepts such as combinatoriality, invariance, and derived rows. Assuming that twelve-tone theory is approached from the grounding in elementary set theory that Mancini proposes, combinatoriality and the derived row may be presented without excessive strain on student mental fortitude. Current twentieth-century textbooks by Joel Lester, Stefan Kostka, and Joseph Straus follow this approach to a greater or lesser extent.[3]

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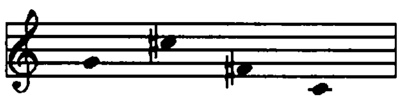
[3]Joel Lester, *Analytic Approaches to Twentieth-Century Music* (New York: W. W. Norton, 1989); Stefan Kostka, *Materials and Techniques of Twentieth-Century Music* (Englewood Cliffs, NJ: Prentice Hall, 1990); Joseph Straus, *Introduction to Post-Tonal Theory* (Englewood Cliffs, NJ: Prentice Hall, 1990).

Invariance proves more slippery; yet it is perhaps most significant from a musical standpoint. Precise tools accessible to the undergraduate student for understanding this concept are lacking. Although the principle of invariance has been examined extensively in the professional literature, many studies brim with algebraic formulae and complex calculations unsuitable for the average undergraduate classroom and are not for the mathematically faint-at-heart.[4] What I propose is a distillation of invariance theory using extensions of existing methods into a form comprehensible to undergraduate students within a limited time, a distillation that propels the student further into the musical intricacies of the twelve-tone universe. I will first examine the basic aspects of set and twelve-tone theory prerequisite to the study of invariance and survey the treatment of invariance in recent textbooks. My proposed method will then be introduced and applied in a brief analysis suitable for classroom presentation.

FIGURE 1. Calculating best normal order

a.



b. G C C# F# (G)

c.	C	C#	F#	G		F#	G	C	C#
	0	1	6	7		0	1	6	7
d.	C	C#	F#	G		F#	G	C	C#
	7	6	1	0		7	6	1	0

[4]The classic studies of invariance include Milton Babbitt, "Twelve-Tone Invariants as Compositional Determinants," *The Musical Quarterly* 46 (1960): 246-259; Donald Martino, "The Source Set and its Aggregate Formations," *Journal of Music Theory* 5 (1961): 224-273; Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973); David Lewin, "A Theory of Segmental Association in Twelve-Tone Music," *Perspectives of New Music* 1 (1962): 89-116; David Beach, "Segmental Invariance and the Twelve-Tone System," *Journal of Music Theory* 20/2 (1976): 157-184; and John Rahn, *Basic Atonal Theory* (New York: Schirmer Books, 1980).

The concepts from set theory that provide a useful foundation for the exploration of twelve-tone theory include pitch class (employing modulo 12 integer notation), interval class, best normal order and set class, and set symmetry. Recognition of symmetrical sets, the most difficult of these concepts to grasp, can be facilitated if approached as a by-product of placing a set in best normal order. Confronted with a set such as that shown in Figure 1a, the student first writes the set in scalar order, beginning on any pitch and repeating the first pitch at the end in parentheses (Fig. 1b). The student next looks for the largest interval between adjacent notes and marks it with a bracket. The instructor should explain the rationale behind this process, that normal order is a referential order that requires that the set be placed in the smallest possible range; we look for the largest interval because it will provide the boundaries of the smallest overall range when inverted. In the example the largest interval appears twice; thus there are two brackets, indicating that the set may be transpositionally symmetrical.

Next the student writes out the set beginning on the pitch(es) on the right end of the bracket(s), thus obtaining the normal order(s) of the set (Fig. 1c). At this point I have the student convert the set into integer notation in order to select best normal order and to identify the set class from a list of prime forms. This conversion is accomplished simply by setting the first pitch to zero and counting ascending half-steps from it. The best normal order likely will be the normal order containing the first smallest interval after the zero. If the normal orders prove to be identical, as in Figure 1, the set is transpositionally symmetrical, meaning that it may be held totally invariant under transposition.

The student is not yet through, however, because of the ticklish possibility that the set is inverted. I find that identifying inverted sets causes more consternation among undergraduate students than any other aspect of set theory. To alleviate this problem I stress that every set has a minimum of two normal orders: one read from left to right and one read from right to left, both spanning the smallest possible range of the set, but doing so from opposite directions.[5] Rather than

[5]The term *normal order* is used here to denote any ordering of a set that encompasses the smallest possible range. *Best normal order* denotes the ordering of the set within the smallest possible range that has the first smallest interval adjacency. This usage follows Forte (1973), 3-5, and differs from *normal form* as used by (among others) Rahn (1980), 31-38. Rahn's normal form is equivalent to Forte's best normal order. In later writings Forte often uses

introducing the complexities of inversion formulae, I merely have them "invert" the process by which they obtained the first set of numerals (Fig. 1d). The last pitch is now set to zero, and the student counts descending half-steps.[6] Now, out of all of the normal orders found through this process (four in the example), the best normal order is the one containing the first smallest numeral after the zero. If the same sequence of numerals appears reading both left to right and right to left, the set exhibits inversional symmetry and will be held totally invariant under inversion followed by transposition.

There are numerous methods and shortcuts for finding best normal order. The method I present here is not entirely original, nor is it my intent in presenting it to endorse it over other methods. I explain it in detail because it is the method the students with whom I have worked have understood and employed most readily, and because the explanation demonstrates that recognizing transpositionally and inversionally symmetrical sets can become an integral component of finding a referential order for a pitch collection. Symmetrical sets are important players in the realm of aspects of twelve-tone theory such as combinatoriality and invariance.

With these fundamentals of set theory in hand, twelve-tone theory may be grasped comfortably. Any of the numerous methods of constructing and labeling a twelve-tone matrix or list of row transformations may be used for an elementary introduction to the system without affecting later study of invariance. Whether the matrix is constructed using integers or pitch names, or whether a list of row transformations in staff notation is used, is irrelevant because, as will be shown below, we will be concerned initially with the intervals between pitches rather than the pitches themselves. In the present discussion, and in my classroom, I use a pitch-name matrix with a "movable zero" designation of row transformations (i.e., the original prime form is labeled "P0" regardless of its starting pitch). Since some students confuse order numbers with transposition numbers if both sets of numbers begin with zero, I thus use 1-12 for order numbers.

After finding the row and constructing some sort of table of row forms, the student can find the row throughout the remainder of the

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normal order to mean *best normal order,* for example in *The Harmonic Organization of 'The Rite of Spring'* (New Haven: Yale University Press, 1978), 3.

[6]Mancini's comments on the musical importance of inversional axes are well taken; the method for recognizing inversion given above is but a first step toward understanding the significance of inversional equivalence. See Mancini, 99-100.

work, labeling row transformations and order numbers. Kostka describes the traditional means of accomplishing this feat:

> Always work from a matrix. If you get lost, try to find several notes that you suspect occur in the same order in some row form, and scan the matrix for those notes, remembering to read it in all four directions.[7]

This method is time-honored, yet also time-consuming. Larry Austin and Thomas Clark offer a more precise method for analyzing serial music in their text *Learning to Compose*.[8] Their "serial-interval" method involves measuring each successive interval in a row according to the number of half-steps it contains when the second note is placed in a higher register than the first note, regardless of the actual relative register of the tones as they appear in the row. To illustrate, we take the row stated at the beginning of Webern's *Quartet*, Op. 22 (Fig. 2a) and write it out in letter names (Fig. 2b; normally the student would do this across the top of a matrix).[9] Next we count the number of half-steps from each pitch up to the following pitch; the specificity created by always counting up gives the system its value.[10] Accordingly, D-flat up to B-flat is nine half-steps, B-flat up to A is eleven, A to C is three, and so forth, continuing as in Figure 2b.

The serial intervals for the inversion of the row obviously represent the inversions of the intervals given in Figure 2b, and may be determined simply by taking mod. 12 complements (Fig. 2c). Because intervals are always reckoned upward, reversing the order of the pitches in retrograde forms also results in interval inversion: starting at the end of the row, G up to D becomes 7 rather than 5. Sharper students quickly notice that the retrograde pattern reverses the inversion pattern (compare 2c and 2d). In like fashion, the serial-interval

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[7]Kostka, 222.

[8]Larry Austin and Thomas Clark, *Learning to Compose: Modes, Materials and Models of Musical Invention* (Dubuque, IA: Wm. C. Brown, 1989). A similar system appears in Straus, 120-125.

[9]That the row stated at the beginning is actually P7 in Webern's original scheme is not of critical importance in the undergraduate classroom. Matrices built on Webern's original "P0" for each of his published works appear in Kathryn Bailey, *The Twelve-Note Music of Anton Webern: Old Forms in a New Language* (Cambridge: Cambridge University Press, 1991), 337-343.

[10]All of the intervals could be measured downward as well, but most students find reckoning upward more intuitive.

FIGURE 2.
Webern, *Quartet*, Op. 22, opening row and serial-interval patterns

a. 

b. P0: Db Bb A C B Eb E F F# G# D G
9 11 3 11 4 1 1 1 2 6 5

c. I0: Db E F D Eb B Bb A G# F# C G
3 1 9 1 8 11 11 11 10 6 7

d. R(P0): G D G# F# F E Eb B C A Bb Db
7 6 10 11 11 11 8 1 9 1 3

e. R(I0): G C F# G# A Bb B Eb D F E Db
5 6 2 1 1 1 4 11 3 11 9

pattern for retrograde inversion reverses that of the prime form (Fig. 2e), and is the inverse of the retrograde pattern.

The serial-interval method proves most useful when the row proves hardest to follow. For example, a pair of intersections in m. 22 of the first movement of Op. 22 often throws students off the track initially. If they look in the next measure, however, they find the pitches B, B-flat, and D stated in succession in the same voice. Analyzing the serial-interval pattern of this fragment yields 11-4. Comparison of this fragment with the catalogue of serial-interval patterns in Figure 2 reveals that the fragment represents the fourth through sixth notes of a prime form (the only appearance of an 11-4 segment). The specific prime form can be deduced by counting half-steps up from C of P0 to the B that occupies the same position in the fragmented row: C to B is eleven half-steps, therefore the fragment comes from P11. The row can be identified much more quickly through this method than by scanning the matrix for the pitches B, B-flat, and D. As I will demonstrate below, the serial-interval pattern also provides a powerful tool for analyzing ordered invariance within a given row.

Each of the three recent textbooks devoted to twentieth-century theory cited above contains a general discussion of invariance, yet none provides a systematic method for predicting and locating it. Lester devotes an entire chapter to "common elements" in the twelve-tone system, and includes a useful summary of the nature and function of invariance:

> Common elements add focus to a passage, bringing a small number of pitch-classes or intervals to the fore, and highlighting them against the background pitches. These highlighted pitch-classes and intervals then form networks with one another, connecting phrases and larger sections to one another.[11]

Although he notes that an interval that occurs more than once in a row will be held invariant in some other transformation of the row, Lester's suggestions for finding common elements, while laudable in their emphasis on musically sensitive listening, fall short of a systematic method:

> In your study of a new piece, begin by listening for prominent pitches and repeated pitch-classes, intervals, and pitch-class sets. As you discover a common element in the piece, try to figure out how it works by finding that common element in a very simple series such as a chromatic scale. This will help increase your understanding of the piece, of the composer's choices among the many precompositional possibilities, of the composer's style, and, finally, of the twelve-tone system itself.[12]

This last sentence provides a worthy answer for the question "What do we want students to learn about this music?" In Lester's approach, however, the burden for discovery rests heavily on student intuition.

Kostka also stresses the importance of invariance for the compositional process: "One of the more difficult tasks of the analyst is attempting to determine why a particular row form and transposition have been chosen. . . .Invariance is frequently a factor."[13] Kostka comes closer to a systematic method, suggesting that the student analyze the interval class between adjacent members of a row and noting that sym-

[11]Lester, 190.
[12]Lester, 203.
[13]Kostka, 218.

metrical patterns within the interval-class succession "may indicate that a segment of the row is its own retrograde or retrograde inversion."[14] The advantage of the serial-interval method over using interval classes resides in the greater accuracy and specificity it provides. A repeating set of adjacent numerals within the serial interval-pattern predicts invariant ordered segments created by a specific operation (transposition or inversion), whereas a repeating set within an interval-class array could indicate invariance by either transposition or inversion.

Straus advocates using the serial-interval pattern to analyze twelve-tone music in general, but he does not apply it in his discussion of invariance. His text, the most technical of the three, uses pitch class to demonstrate how dyads may remain invariant under inversion, with this demonstration coming from almost a compositional point of view:

> Let's say we have within a series a dyad of which we are particularly fond. We are so fond of this dyad that we want to be sure to keep it as a unit even while we transform the series. It's not hard to do, as long as there is an equivalent dyad elsewhere in the series. We simply choose a T_nI that will map those two dyads onto one another. . . . Let's try to map [(0,11)] onto (6,5). . .We are looking for the index number of [11,0] and [5,6]. The index number is 5 (0 + 5 = 6 + 11). T_5I will thus map 0 and 11 onto 6 and 5 (and vice versa).[15]

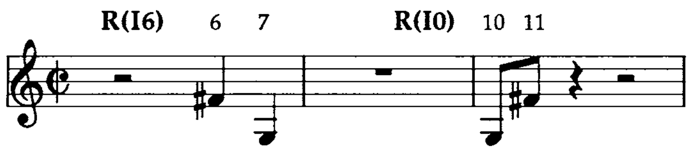
Straus notes that the same principles apply to holding larger collections invariant, although he does not provide a complete exposition of these principles nor a systematic means of finding both ordered and unordered invariance.

Analysis of the row in Webern's *Das Augenlicht*, Op. 26 will prove the usefulness of the serial-interval method for predicting ordered invariance. Invariance in and of itself, of course, is not always a significant phenomenon. For example, two-note invariant segments are inevitable; at least one of each of the eleven dyad adjacencies within the row will be held invariant in at least one inverted row transformation. Furthermore, any repeated interval class within the adjacent pitch classes of a row creates an invariant dyad segment under both transposition and inversion. Since there are only six possible interval

[14]Kostka, 214.

[15]Straus, 147-148.

EXAMPLE 1. Webern, *Das Augenlicht*, Op. 26, Violin, mm. 4-6



Webern *Das Augenlicht*, Op. 26

classes, and eleven adjacencies within a row, repeated interval classes always exist. These facts deny invariant dyads any intrinsic significance. They attain importance only if represented several times within the row or if manipulated so that they are especially prominent musically.[16]

Webern frequently emphasizes invariant dyads through registral or timbral isolation.[17] *Das Augenlicht* opens with a seven-bar instrumental introduction that presents interlocking statements of R(I0), R(I6), and R(P5). The violin part for these measures contains only four notes (see Ex. 1): the F-sharp and G in m. 4 are the sixth and seventh notes of R(I6), and the G and F-sharp in m. 6 are the tenth and eleventh notes of R(I0). Webern thus highlights the invariant dyad through both register and timbre.

Invariant segments larger than the dyad are not inevitable, and thus possess greater intrinsic significance. Larger segments can be located by comparing all forty-eight row transformations or, as Lester suggests, by careful listening. The serial-interval method allows the presence of invariant segments to be deduced directly from the original row before even constructing a matrix simply by locating repeating patterns among the numerals of the serial-interval pattern. Figure 3a contains the prime form of the *Das Augenlicht* row, the transposition levels that would be written across the top of the matrix, and the row's serial-interval pattern. Reading from left to right, the 4-1

[16]Invariant dyads are discussed at length in Babbitt, "Twelve-Tone Invariants as Compositional Determinants."

[17] See especially his *Variations*, Op. 27.

pattern found as the fifth and sixth numerals repeats as the tenth and eleventh numerals. This repeated pattern will create an invariant trichord segment under transposition. The interval-class difference of the pitch classes beginning each appearance of the segment provides the transposition operators that will realize the invariant trichords: interval class 3 separates B (3) and D (6), and therefore any two row transformations related to one another as P0 relates to P3 or as P0 relates to P9 will share an invariant ordered trichord (Fig. 3, b and c).[18]

As shown in Figure 4a, a second invariant trichord occurs toward the end of the row: the 9-4 pattern found as the seventh and eighth numerals repeats as the ninth and tenth numerals. The interval-class difference of the E (8) and F (9) that begin these patterns is 1; there-

FIGURE 3. *Das Augenlicht,* row and invariant trichords

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
		2	11	3	11	4	1	9	4	9	4	1
b. P3:	B	C#	C	Eb	D	F#	G	E	G#	F	A	Bb
c. P9:	F	G	F#	A	G#	C	C#	Bb	D	B	Eb	E

FIGURE 4. Invariant trichords under transposition

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
		2	11	3	11	4	1	9	4	9	4	1
b. P1:	A	B	Bb	C#	C	E	F	D	F#	Eb	G	G#
c. P11:	G	A	G#	B	Bb	D	Eb	C	E	C#	F	F#

[18]The P0-P3 and P0-P9 relationships are abstractly identical, but it is helpful (especially to undergraduates) to list every row transformation that shares invariant segments with P0.

fore any two row transformations related to one another as P0 relates to P1 or as P0 relates to P11 will share an invariant ordered trichord (Fig. 4, b and c).

Invariant ordered segments that appear under retrograde inversion may also be deduced directly from the original serial-interval pattern. Because the process for determining the transposition level is a bit more complicated for these segments, however, I prefer to introduce invariant segments first under simple retrograde. These are found by comparing the original serial-interval pattern read left to right with the inverted serial-interval pattern read right to left (Fig. 5). The 3-11 segment occurring as the third and fourth numerals of the original pattern appears in reverse order as the sixth and seventh numerals of the inverted pattern. Notice that the corresponding 1-9 and 9-1 patterns also match; this numeric exchange will always occur. Determining the transposition level of the retrograde containing the invariant trichord again can be accomplished by consulting the pitch classes of P0, but this time we must compare the pitch classes on opposite ends of the respective trichords. Interval class 4 separates A (the first pitch of the first trichord) and C-sharp (the last pitch of the second trichord); therefore any row transformation related as P0 to R(P4) or P0 to R(P8) will share an invariant ordered trichord (Fig. 5, b and c).

FIGURE 5. Invariant trichords under retrograde

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
		2	11	3	11	4	1	9	4	9	4	1
Inv:		10	1	9	1	8	11	3	8	3	8	11
b. R(P4):	B	Bb	F#	A	F	G#	G	Eb	E	C#	D	C
c. R(P8):	Eb	D	Bb	C#	A	C	B	G	G#	F	F#	E

Any segment of the inverted serial-interval pattern that duplicates a segment of the original pattern when both segments are read from left to right indicates an invariant ordered set under inversion. In Figure 6a, the 11-3 segment between the second and third numerals of the original pattern repeats as the sixth and seventh numerals of the inversion, creating yet another invariant trichord. Notice again that the segments in the corresponding positions also match.

Deducing the transposition level of the inversion that contains the invariant segment requires a different calculation from that used for transposition and retrograde. For this calculation the pitch-class numerals written above the row become especially handy. Find the pitch-class numeral above the first note of the first appearance of the segment and add this value to the pitch-class numeral appearing over the first note of the second appearance of the pattern; this process yields the transposition level of the inversion that shares the invariant segment. In the example, the first note of the first segment is B-flat, pitch class 2, and the first note of the second segment is E-flat, pitch class 7; 7 plus 2 equals 9, and therefore any row transformations related as P0 to I9 will share two invariant trichord segments (Fig. 6b). A special property of inversion is that the first segment will always map onto the position of the second segment and vice versa.

FIGURE 6. Invariant trichords under inversion

0 2 1 4 3 7 8 5 9 6 10 11
a. P0: G# Bb A C B Eb E C# F D F# G
2 11 3 11 4 1 9 4 9 4 1

Inv: 10 1 9 1 8 11 3 8 3 8 11

b. I9: F Eb E C# D Bb A C G# B G F#

As mentioned before, the presence of invariant segments under retrograde inversion can be deduced directly from the original serial-interval pattern: any symmetrical pattern of numerals (reading the same from left to right or right to left) will generate invariant segments under retrograde inversion. Three such segments exist in the *Das Augenlicht* row (see Fig. 7a): 11-3-11, 9-4-9, and 4-9-4. The process for finding the specific retrograde inversion resembles that for finding shared segments under inversion alone: add the pitch-class values of the first and last notes of the symmetrical pattern. For the 11-3-11 segment, 2 (B-flat) plus 3 (B) makes 5;therefore any row transformations related as P0 to R(I5) will share an invariant ordered tetrachord (Fig. 7b). For the 9-4-9 segment, 8 (E) plus 6 (D) makes 14. Since we are using mod. 12 addition, 12 is subtracted from any number greater than 11. Therefore, any transformations related as P0 to R(I2) will share an invariant ordered tetrachord (Fig. 7c). For the 4-9-4 segment, 5 (C-sharp) plus 10 (F-sharp) makes 15, 15 equals 3 mod. 12, and the invariant tetrachord will be realized in R(I3) (Fig. 7d).

FIGURE 7. Invariant tetrachords under retrograde inversion

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
	2	11	3	11	4	1	9	4	9	4	1	
b. R(I5):	D	Eb	G	E	G#	F	F#	Bb	A	C	B	C#
c. R(I2):	B	C	E	C#	F	D	Eb	G	F#	A	G#	Bb
d. R(I3):	C	C#	F	D	F#	Eb	E	G#	G	Bb	A	B

FIGURE 8. Invariant segments under retrograde inversion

0	2	3	4	5	6	7	1	8	9	11	10
A	B	C	C#	D	Eb	E	Bb	F	F#	G#	G
2	1	1	1	1	1	6	7	1	2	11	

In the *Das Augenlicht* row the elements of each symmetrical pattern are contiguous; this is not always the case. In the serial-interval pattern in Figure 8 the 2-1 . . . 1-2 grouping forms a symmetrical pattern across the intervening numerals. Any segment of the serial-interval pattern that repeats in reverse will create ordered invariance under retrograde inversion. Successions of repeated numerals also form symmetrical patterns, such as the 1-1-1-1-1 pattern in Figure 8 that predicts an invariant ordered hexachord under retrograde inversion. The process for finding the transposition level of the retrograde inversion sharing the invariant segments remains the same in either case.

The serial-interval method provides a systematic means of finding invariant ordered segments that does not burden undergraduate students with complex concepts or formulae, requiring only that they be able to calculate interval sizes in half-steps and be able to employ mod. 12 addition. The students now possess a tool that can be applied easily in analysis or composition. The search for unordered invariance requires a different tool, since it cannot be deduced solely from the ordered serial-interval pattern. Unordered invariant sets are unearthed through the process of imbrication, or taking each possible three-,four-, five-, and six-note segment of adjacent pitches and placing them in best normal order. Although a "brute force" method that lacks the elegance of the serial-interval pattern, imbrication nonetheless produces results without demanding new skills.

The two situations indicating unordered invariance are (1) the repetition of a set class within the row and (2) the presence of a symmetrical set. Any set class that appears more than once will be held invariant under transposition as long as both appearances of the set are in prime, or both in inverted, form; if only one appearance of the set is inverted, the set will be held invariant under inversion. Figure 9 lists the five-note sets contained in the *Das Augenlicht* row; in order to keep track of inverted sets, I have students mark them with a lower-case "i" (placed at the end). Set class 5-1 appears twice in the row; because 5-1 is inversionally symmetrical, these two appearances may represent two prime forms or two inversions, or a prime form and an inversion. We will take the first possibility first, and save the other one for the discussion of symmetrical sets.

FIGURE 9. Pentachords in the row of *Das Augenlicht*

Pentachords	Prime Form	Set Class
G#, A, Bb, B, C	0,1,2,3,4	5-1
A, Bb, B, C, Eb	0,1,2,3,6	5-4
A, B, C, Eb, E	0,1,4,5,7 i	5-Z18 i
B, C, C#, Eb, E	0,1,2,4,5	5-3
B, C#, Eb, E, F	0,1,2,4,6 i	5-9 i
C#, D, Eb, E, F	0,1,2,3,4	5-1
C#, D, E, F, F#	0,1,2,4,5 i	5-3 i
C#, D, F, F#, G	0,1,2,5,6 i	5-6 i

The process for finding the transposition level that realizes the invariant unordered pentachord resembles that used for ordered subsets, differing only in that the first notes of the best normal orders are compared rather than the first notes of the segments in actual row order. The difference of the pitch-class numbers for these pitches as they appear in the row still provides the transposition level. As shown in Figure 10a, G-sharp (0) and C-sharp (5) differ by interval class 5; therefore any transformations related as P0 to P5 or P0 to P7 will share an invariant unordered pentachord (Fig. 10, b and c).

Set class 5-3 also appears twice in Figure 9, once in prime and once in inverted form, thus being held.invariant under inversion. Inverted sets can cause difficulties for students because they must be read in descending order; in other words, the pitch representing "0" in the best normal order for the inverted 5-3 (C-sharp,D, E, F, F-sharp) is F-sharp, not C-sharp. The process for predicting which inverted row transformation holds an unordered set invariant resembles the method used for invariant ordered segments under inversion, although again using the first notes of the best normal order rather than actual row order. The pitch-class value of the first note of the best normal order of the first appearance of the set class is added to the pitch-class value of the first note of the best normal order of the second appearance of the set class. In Figure 9, 5-3 begins with B (3) and 5-3i with F-sharp (10); 3 plus 10 makes 1 (mod. 12). Therefore, any row transformations related as P0 to I1 will share two unordered pentachords (Fig. 11). In this particular instance the two pentachords interlock to form an invariant eight-note set.

FIGURE 10. Invariant unordered pentachords under transposition

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
b. P5:	C#	Eb	D	F	E	G#	A	F#	Bb	G	B	C
c. P7:	Eb	F	E	G	F#	Bb	B	G#	C	A	C#	D

FIGURE 11. Invariant unordered pentachords under inversion

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
b. I1:	A	G	G#	F	F#	D	C#	E	C	Eb	B	Bb

The presence of symmetrical sets also creates unordered invariance, with transpositionally symmetrical sets being held invariant under transposition and inversionally symmetrical sets being held invariant under inversion. Transpositionally symmetrical sets divide into two groups: those with equidistant elements and those without. The equidistant sets ([0,6], [0,4,8], [0,3,6,9], and [0,2,4,6,8,10]) hold invariant when transposed by any interval class they contain: [0,6], holds invariant in P6, [0,4,8] holds invariant in P4 and P8, [0,3,6,9] holds invariant in P3, P6, and P9, and [0,2,4,6,8,10] holds invariant in P2, P4, P6, P8, and P10. Transpositionally symmetrical sets whose elements are not equidistant hold invariant under tritone transposition (P6) except for [0,1,4,5,8,9] (set class 6-20), which holds invariant in P4 and P8.[19] The *Das Augenlicht* row contains no transpositionally symmetrical subsets larger than the two-element tritone, which cannot be unordered in the context of a twelve-tone row.

The transposition levels of the inverted row transformation that will hold an inversionally symmetrical set invariant is determined by taking the first and last pitches of the best normal order and adding their pitch-class values in the original row. The inversionally symmetrical pentachord 5-1 appears twice in the *Das Augenlicht* row. The first and last pitches of the best normal order for the first appearance are G-sharp and C (Fig. 9), the sum of their pitch-class values in the original row (0 and 4) is 4, and therefore any row transformations related as P0 to I4 share an invariant unordered pentachord (Fig. 12, a and b). The best normal order for the second appearance of 5-1 begins on C-sharp (5) and ends on F (9), which sum to 2 (mod. 12); any row transformations related as P0 to I2 share an invariant unordered pentachord (Fig. 12c).

The 5-1 pentachord engenders still more invariant relationships in addition to the transpositional invariance demonstrated in Figure 10 and the inversional invariance generated by its symmetry demonstrated in Figure 12. Because it exhibits inversional symmetry, and because it appears twice in the row, one appearance may act as a prime form and the other as an inversion, thus creating additional invariance under inversion. Taking the first appearance as the prime form

[19]This refers only to hexachords and smaller sets. The complements of these transpositionally symmetrical sets are themselves transpositionally symmetrical and hold invariant under the same transposition levels. For a complete listing and examination of transpositionally symmetrical sets, see Richard Cohn, "Properties and Generabilities of Transpositionally Invariant Sets," *Journal of Music Theory* 35 (Spring/Fall 1991): 1-32.

FIGURE 12. Invariant unordered pentachords under inversion (inversionally-symmetrical set)

	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
b. I4:	C	Bb	B	G#	A	F	E	G	Eb	F#	D	C#
c. I2:	Bb	G#	A	F#	G	Eb	D	F	C#	E	C	B

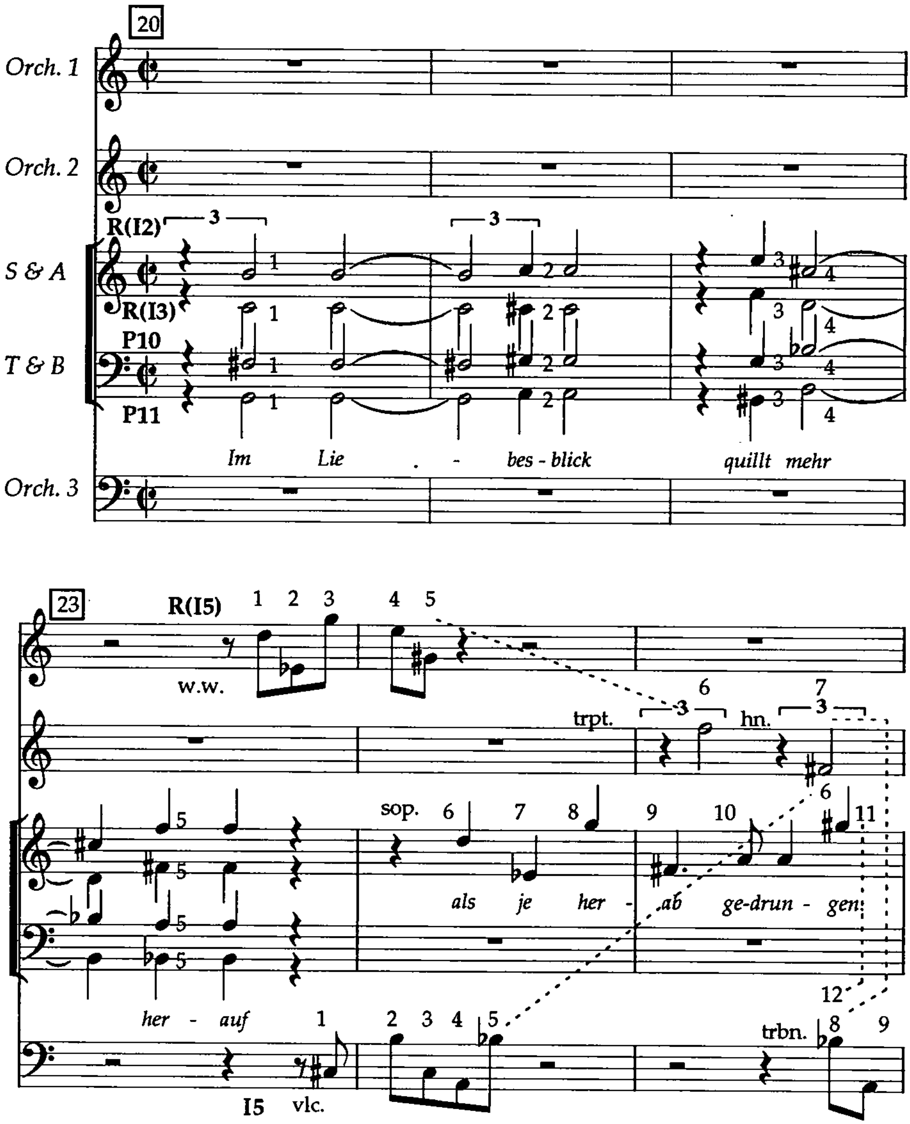
FIGURE 13. Invariant unordered pentachords under inversion (inversionally-symmetrical set)

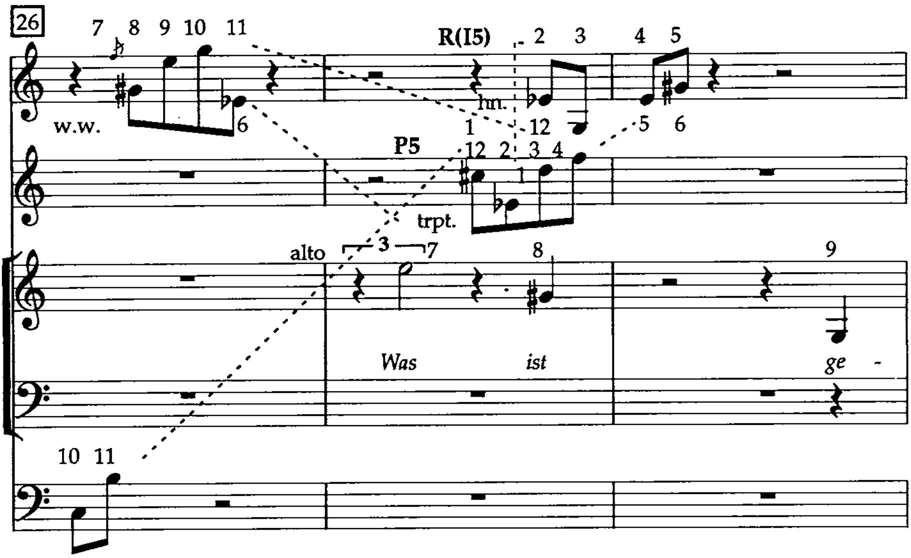
	0	2	1	4	3	7	8	5	9	6	10	11
a. P0:	G#	Bb	A	C	B	Eb	E	C#	F	D	F#	G
b. I9:	F	Eb	E	C#	D	Bb	A	C	G#	B	G	F#

and the second as the inversion (see Fig. 9), we add the pitch-class values of G-sharp (0) and F (9) to obtain the specific inversion that contains the invariant sets (Fig. 13). Any row transformations related as P0 to I9 will share two invariant unordered pentachords, again due to the reciprocal mapping that takes place under inversion.

The row to *Das Augenlicht* possesses a significant amount of both ordered and unordered invariance, and examination of a passage from the work will demonstrate that this invariance heavily influenced Webern's choice and use of specific row transformations. Example 2 contains the excerpt in short score, with all pitches sounding as written. The four choral voices share the same rhythm at the beginning of the excerpt, causing a strict note-against-note arrangement of four row transformations. The types of row transformations used and their transposition levels suggest soprano-alto and tenor-bass pairings. The continuation of the passage supports these pairings, since the soprano and alto lines conclude independently while the tenor-bass pair continues only when new row transformations enter in the upper voices in m. 30. The R(I2)-R(I3) pairing in the soprano and alto, being equiva-

EXAMPLE 2. Webern, *Das Augenlicht,* Op. 26, mm. 20-37




26
7 8 9 10 11
R(I5)
2 3
4 5
w.w.
6
hn.
1
12
5 6
P5
12 2 3 4
1
trpt.
alto
3
7
8
9
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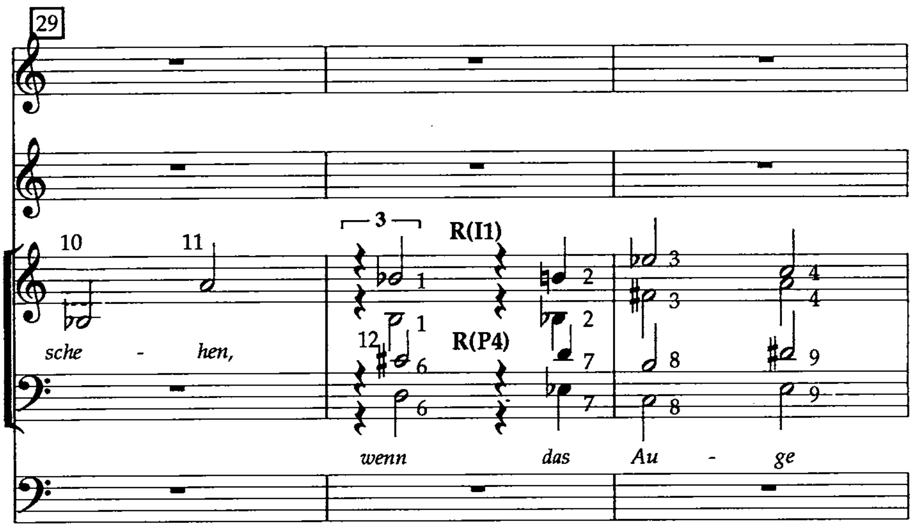
ist
ge -
10 11


29
3
R(I1)
10
11
1 2 3 4
12
R(P4)
6 7 8 9
sche - hen,
wenn das Au - ge

EXAMPLE 2. (continued)

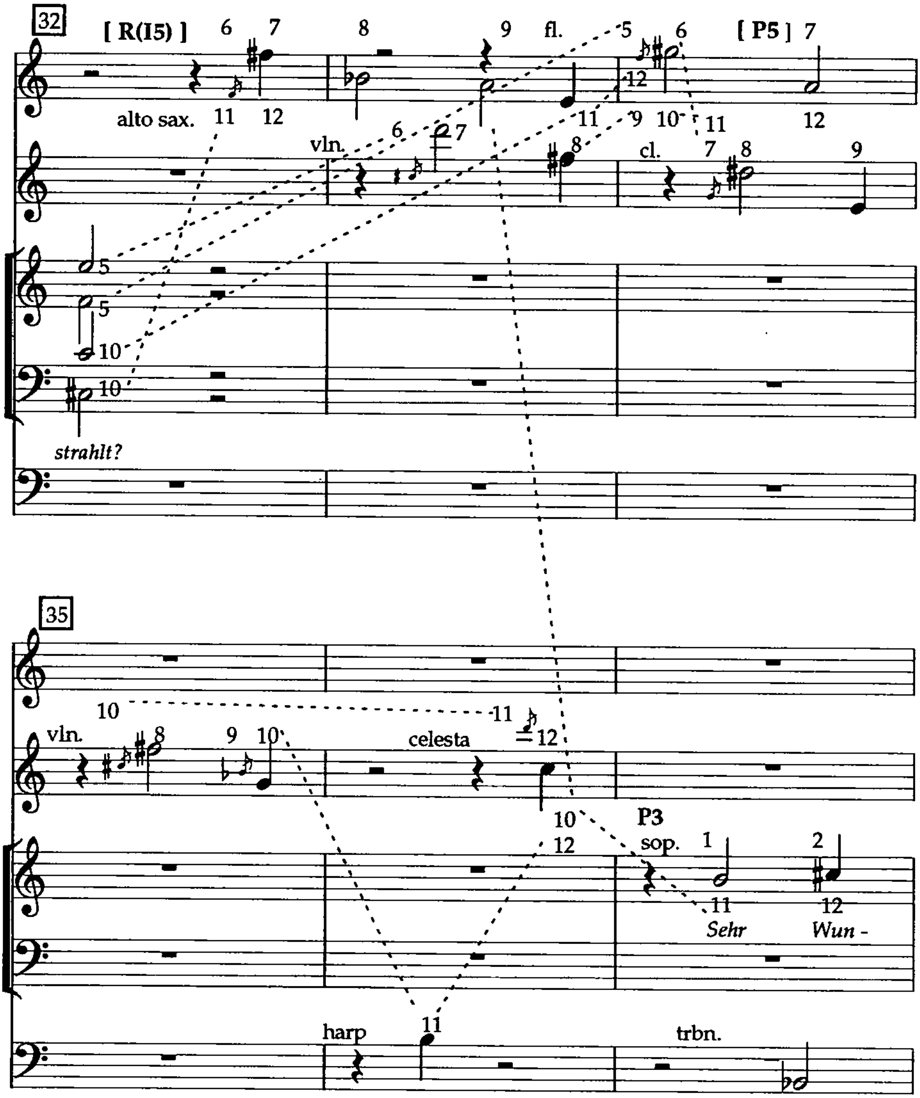


FIGURE 14. Assymetrical combinatoriality

R(I2):	B C E C# F D Eb	G F# A G# Bb
P10:	F# G# G Bb A	C# D B Eb C E F

lent to the P0-P1 pairing demonstrated in Figure 4, contains an invariant ordered trichord that is realized in mm. 21-24. The C-sharp-F-D in the alto in mm. 21-23 is echoed in the soprano in mm. 22-24. Even though rests intervene in mm. 23-24, Webern emphasizes the invariant trichord by maintaining the same contour. The tenor-bass pair P10-P11 also bears the P0-P1 relationship; the invariant trichord generated by this pairing will be discussed in connection with mm. 30-32.

Invariance also governs the other row pairings at the beginning of the passage. Unordered invariance plays a role in the R(I2)-P10 pairing between the soprano and tenor (equivalent to the P0-I4 pairing in Fig. 12) and the identical R(I3)-P11 pairing in the alto and bass. Each pairing contains an invariant pentachord and an invariant septachord that appear at opposite ends of the respective row transformations, resulting in a type of asymmetrical combinatoriality that prevents octave duplications (see Fig. 14; the row is non-combinatorial in the traditional sense). The R(I3)-P10 pairing between the alto and tenor, being equivalent to P0-R(I5) (Fig. 7b), shares an invariant ordered tetrachord. The G-sharp-G-B-flat-A in the tenor in mm. 21-23 reappears with a different contour in the alto in mm. 27-29. The final pairing, grouping R(I2) in the soprano with P11 in the bass, is equivalent to the P0-R(I3) pairing of Figure 7d and thus also shares an invariant ordered tetrachord. The C-E-C-sharp-F in the soprano in mm. 21-23 reappears in the bass and alto sax in mm. 31-32.[20]

The orchestra enters with the retrograde-related pair I5-R(I5) at the cessation of the block chords in m. 23. Webern presents the first five notes of each instrumental row form as a discrete motive, reflecting the five-note segmentation of the preceding chordal passage. While the retrograde relationship obviously controls the pairing of I5 and

[20]This is the only pairing that creates octave duplication. The potential octave occurs in the sixth position of the row transformations, precisely after the point at which Webern breaks off the strict note-against-note writing in m. 23.

R(I5), the principle of invariance led Webern to link this pair to the continuation of R(I2) in the soprano line in m. 24: R(I2) and R(I5) share an invariant ordered trichord by virtue of transposition by interval class 3 (as demonstrated in Figure 3). The invariant trichord stands out clearly: the D-E-flat-G beginning the woodwind motive in m. 23 reappears at the same pitch level at the resumption of R(I2) in the soprano in m. 24.

The alto's R(I3) also interacts with the I5-R(I5) pair. The last two pitches of the first statement of the invariant trichord in m. 23 (E-flat and G) join with the following two woodwind pitches (E and G-sharp) to form an invariant tetrachord that permeates mm. 26-28. Using retrograde-related row transformations obviously provides ample opportunity for invariance, such as the five-note cello motive in mm. 23-24 that returns in retrograde in mm. 25-27; Webern goes beyond the obvious, however, by linking the continuation of R(I3) in the alto with the I5-R(I5) pair through the unordered tetrachord. The tetrachord appears in the woodwinds in m. 26 in the order G-sharp-E-G-E-flat, and again in mm. 27-28 in retrograde: E-flat-G-E-G-sharp. The trumpet introduces P5 in m. 27, and the fifth and sixth notes of this row intersect with the E and G-sharp ending the retrograde statement of the invariant tetrachord in m. 28. Another intersection occurs in m. 26, where the E-flat in the woodwinds serves as the sixth pitch of the alto's R(I3) and introduces the invariant tetrachord in the order E-flat-E-G-sharp-G. Finally, the B-flat-A dyad in the alto in m. 29 duplicates the dyad which ends the five-note motive in m. 24 and which begins its retrograde in m. 25.

A second chordal passage begins in m. 30, retaining the five-note length of the first passage and maintaining the same contour in the soprano line. The tenor and bass resume P10 and P11 after six bars of rest, and the invariant trichord they share is partially represented: the E-flat-C-E in the bass in mm. 30-31 is begun in the tenor in mm. 31-32. The chordal passage ends before the trichord concludes; its last tone appears in the flute in m. 33. Webern presents a new pairing between the soprano and alto: their R(I1)-R(P4) pairing is equivalent to the P0-I9 pairing illustrated in Figures 6 and 13, and thus shares two invariant ordered trichords as components of two invariant unordered pentachords. The first invariant trichord (E-C-sharp-D) is not well delineated in the music; the second (F-G-sharp-G) appears as a literal intersection in the orchestra in m. 34. Invariance between R(I1) and R(P4) that occurs entirely within the chordal passage is limited to the opening dyad: the soprano and alto merely exchange notes in m. 30.

The intersection of invariant segments becomes quite prevalent when R(I1) in the soprano interacts with P10 and P11 in the tenor and bass. R(I1) and P10 share an ordered tetrachord; the second through fifth notes of the soprano in mm. 30-32 appear in stretto imitation one note later in the tenor (the last note appears in the flute in m. 33). At the same time, R(I1) and P11 share an ordered tetrachord that intersects with the one shared by R(I1) and P10. The E-flat-C-E-C-sharp in the bass line of mm. 30-32 appear in stretto imitation one note later in the soprano, with the final C-sharp appearing in the violin. Figure 15 illustrates the intersection of the various invariant segments operative in mm. 30-32. The trichord E-flat-C-E appears three times, first in P11, then R(I1), and finally P10, with each overlapping entrance retaining the same contour. The six row transformations active at the end of the second chordal passage complete their courses with numerous intersections in mm. 32-36.

Invariance, both ordered and unordered, clearly governed Webern's choice and use of row forms in the cited passage.[21] Every row pairing used simultaneously exhibits one or more invariant seg-

FIGURE 15. Intersection of invariant segments, *Das Augenlicht* mm. 30-32

R(I1):	Bb 1	B 2	Eb 3	C 4	E 5	(C#) 6
R(P4):	B 1	Bb 2	F# 3	A 4	F 5	(G#) 6
P10:	C# 6	D 7	B 8	Eb 9	C 10	(E) 11
P11:	D 6	Eb 7	C 8	E 9	C# 10	(F) 11

[21]In light of the analysis presented above, Kathryn Bailey's comments on page 23 concerning *Das Augenlicht* seem curious: "Although the Op. 26 row offers considerable invariant combinations, Webern does not take advantage of this property; therefore it is not significant."

ments, and successive row transformations often intersect on invariant beginning and ending pitches. An analysis that merely traces the row through the work misses the emphasis Webern places on certain pitches and pitch collections through invariance, and therefore misses the musical (not mathematical) intricacy of the passage.

The methods outlined in this paper not only provide systematic tools for predicting and locating invariance, but also provide at least an elementary appreciation of how invariance works and how it might influence the musical structure of a twelve-tone work. Just as the construction of the row precedes composition of a twelve-tone work, analysis of the properties of the row should precede analysis of the work. By first analyzing the row itself, the analyst can better understand the precompositional and compositional processes of the composer. The degree of ordered invariance the student might encounter ranges from totally invariant symmetrical rows, such as those in Webern's *Symphony*, Op. 21 (Fig. 16a) and Dallapiccola's *Cinque Canti* (Fig. 16b), to rows that contain no invariant ordered segment larger than the dyad, such as that in Dallapiccola's *Quaderno Musicale di Annalibera* (Fig. 16c). Most rows fall between these extremes of ordered invariance. Unordered invariant sets larger than the dyad are more

FIGURE 16. Selected rows

a. Webern, *Symphony*, Op. 21.

P0:	F	Ab	G	F#	Bb	A	Eb	E	C	C#	D	B
	3	11	11	4	11	6	1	8	1	1	9	
R(P):	3	11	11	4	11	6	1	8	1	1	9	

b. Dallapiccola, *Cinque Canti.*

Gb	F	B	D	C	Ab	C#	A	G	Bb	E	D#
11	6	3	10	8	5	8	10	3	6	11	

c. Dallapiccola, *Quaderno Musicale di Annalibera.*

	Bb	B	Eb	Gb	Ab	D	Db	F	G	C	A	E
	1	4	3	2	6	11	4	2	5	9	7	
Inv:	11	8	9	10	6	1	8	10	7	3	5	

ubiquitous, and will usually be utilized at some point in the work.

A word of warning: students should not be disappointed if they find a row rich in invariance that was not exploited by the composer, or that was used only in part. They have discovered that the composer must be up to something else, a discovery that should prompt them to search for other reasons behind a composer's choice of row transformations. For example, the serial-interval pattern of the row to Webern's *Quartet*, Op. 22 (given in Fig. 2) contains two symmetrical patterns that yield ordered tetrachords by retrograde inversion, both of which appear in the pairing P0-R(I7). Throughout the cited movement, however, Webern pairs primes and inversions whose transposition levels sum to 10 (P0-I10, P6-I4, P9-I1, etc.) in order to retain the same set of dyad pairs between row transformations.[22] Although the P0-R(I7) pairing (or its equivalent) with its invariant ordered tetrachords does not appear, other unordered invariant groupings resound throughout.

After a grounding in basic set and twelve-tone theory as set forth above, the concept of invariance and the systematic method for finding it can be presented in one or two class sessions. Another one or two sessions should be spent leading the class through several analytical applications of the method before requiring them to apply it on their own. As a final assignment before concluding the unit on twelve-tone music, I suggest having the students write their own twelve-tone works employing the concepts studied in the unit. I ask my students to construct a row, being aware of its properties as they do so, and then manipulate these properties in a sixteen- to thirty-two bar composition. An effective variant of this assignment involves giving all of the students the same row and instructing them to write a brief composition utilizing its properties. Take a class period or two and have the students report on their compositions (what worked well, what did not), or select a few representative compositions and lead the class in a discussion of their most interesting features. Although the result will seldom be great art, students can become engrossed in the project and discover facets of the twelve-tone system far beyond those normally presented in the undergraduate classroom. Through the methods proposed in this paper students can begin to see the music beyond the math.

[22]See George Perle, "Webern's Twelve-Tone Sketches," *The Musical Quarterly* 57/1 (January, 1971): 1-25.