Recommendations for Atonal Music Pedagogy in General; Recognizing and Hearing Set-Classes in Particular

Robert D. Morris

Introduction

As its title indicates, this paper is directed to those who presently teach or are planning to teach atonal[1] music theory in any number of pedagogical settings: graduate composition, analysis, or theory classes and/or seminars; twentieth-century music literature classes; undergraduate theory sequences; workshop/seminars introducing compositional techniques and the like. My objective is first to raise several issues that address the role of atonal music theory in these highly different contexts, then to provide some suggestions regarding terminology and notation that will be useful in most educational occasions.

In the second part of the paper, I outline a methodology to teach the written and aural identification of set-classes and transformations. New and older concepts of set-class categorization and transformation are applied to this end. The final goal involves committing all or at least a large proportion of the hexachordal set-classes to memory and identification, in both visual and aural modalities. To this end, I introduce a simple and musically relevant generative method that doesn't rely on tedious calculation, table look-ups, or sheer rote memory.

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[1]I use the word "atonal" without bias simply to refer to twentieth-century music whose structure is based on relations of pitches and pitch-classes that are independent of tone centers and/or functional harmony.

The Field of Atonal Music Theory

Atonal music theory and analysis address the study of nontonal Western art music of the twentieth century. Included are the well-known topics of "set theory" and "classical" twelve-tone theory (and its extensions such as generalized combinatoriality), as well as more idiomatic approaches (such as George Perle's "Twelve-Tone Tonality") or those theories that do not isolate atonal and tonal issues in principle or practice (like those developed in David Lewin's *Generalized Intervals and Transformations*).[2] The field, like many other academic disciplines, has developed "top-down" from the sophisticated and advanced concerns of articulate composers and theorists to its recent incorporation in undergraduate music curricula. But for all its present day unification and generality, atonal music theory's development originated in a nonlinear manner, in a few minimally overlapping "schools" of highly contrasting concerns. A brief review of the history of the field will help pinpoint some of this characteristic diversity.

Research has confirmed that much of the important music of the twentieth century can be modeled (at least in part) with the tools of atonal music theory. The structure of music by Claude Debussy, Alexander Skryabin, Arnold Schoenberg, Alban Berg, Anton Webern, Ernst Krenek, Edgard Varèse, Igor Stravinsky, Béla Bartók, Olivier Messiaen, Elliott Carter, Pierre Boulez, Milton Babbitt, and a host of other composers has been illuminated by such tools, whether or not the composer consciously applied any kind of pitch/pitch-class theory to his or her works. Moreover, composers such as Messiaen, Schoenberg, Howard Hanson, Babbitt, Donald Martino, George Perle, Daniel Starr, and Robert Morris, to mention only a few, have published various compositional theories of pitch/pitch-class, many of which are general and flexible enough to be useful to other composers and to stimulate theorists and analysts who do not compose.[3]

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[2]See George Perle, *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977); David Lewin, *Generalized Musical Intervals and Transformations* (New Haven and London: Yale University Press, 1987).

[3]See Olivier Messiaen, *Technique de mon langage musical* (Paris: Leduc, 1944); Arnold Schoenberg, "Composition with Twelve Tones" in *Style and*

But despite its general interest to scholars in all walks of music, the work of composer/theorists is often more oriented to making music than to analyzing it; indeed, some composers (such as John Cage, Karlheinz Stockhausen, or Iannis Xenakis[4]) have invented compositional methods that are meant to affect their music's character or flow only globally, or simply to provide new ways of making interesting music.[5] While such theories may be highly influential among composers, they do not provide much substance for analysts interested in producing descriptions of how pieces go.

The rise of music theory in the 1960s as an autonomous musical discipline distinct from musicology or composition is marked by important studies of the structure and analysis of nontonal music.[6]

Idea: Selected Writings of Arnold Schoenberg, ed. Leon Stein (New York: St. Martins Press, 1975); Howard Hanson, *Harmonic Materials of Modern Music* (New York: Appleton-Century-Crofts, 1960); Milton Babbitt, "Some Aspects of Twelve-Tone Composition," *The Score and IMA Magazine* 12 (1955): 53-61; Babbitt, "Twelve-tone Invariants as Compositional Determinants," *The Musical Quarterly* 46 (1960): 245-59; Babbitt, "Set Structure as a Compositional Determinant," *Journal of Music Theory* 5/1 (1961): 72-94; Babbitt, "Twelve-Tone Rhythmic Structure and the Electronic Medium," *Perspectives of New Music* 1/1 (1962): 49-79; Babbitt, "Since Schoenberg," *Perspectives of New Music* 12/1,2 (1973): 3-28; Donald Martino, "The Source Set and its Aggregate Formations," *Journal of Music Theory* 5/2 (1961): 224-73; George Perle, *Twelve-Tone Tonality*; Daniel V. Starr, "Derivation and Polyphony," *Perspectives of New Music* 23/1 (1984): 180-257; Robert D. Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven and London: Yale University Press: 1987).

[4]See for instance Karlheinz Stockhausen, "How Time Passes," *Die Riehe* 3 (1959): 10-40; John Cage, *Silence* (Middletown, CT: Wesleyan University Press, 1961); Iannis Xenakis, *Formalized Music: Thought and Mathematics in Composition* (Bloomington, IN: Indiana University Press, 1971).

[5]Of course, some composers' methods may be intended to be heard as such, but are not, due to flaws in compositional thinking or to the use of such methods in ways that do not engage basic aspects of human psychoacoustical cognition.

[6]This includes George Perle, *Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg, and Webern* (Berkeley: University of California Press, 1962); Allen Forte, "A Theory of Set-Complexes for Music," *Journal of Music Theory* 8/2 (1966): 136-83; Forte, *The Structure of Atonal Music* (New Haven and London: Yale University

Numerous analytic articles and books provide a rich, insightful, and diverse set of "case studies." After a generation of composers and theorists worked on developing and understanding various models of structure in twentieth-century music, the field began to affect undergraduate theory curricula.[7] Nevertheless, that there is only one ear training manual based on current atonal music theory concepts, Friedmann's *Ear Training for Twentieth-Century Music* (1990),[8] indicates the field of atonal music pedagogy is still young.

Distinctions between types of theories that affect pedagogy.

1. *Theories of Composition versus Analysis*

The study of atonal music is modulated by compositional and analytic use. While these two avenues may seem allied, they are actually quite different. A composer is interested in making his or her music original, even unique, suggesting that compositional theory is speculative. The primary concern is for general, open-ended principles of composition, not simply descriptions of how existing pieces are structured. On the other side, the analyst is not necessarily interested in how pieces might be made, but in how existing pieces can be construed. In the analyst's quest for appreciation of music's structural detail and its contribution to the style and essence of a particular piece, the use of theory may become too specialized for generalization to other pieces, much less to pieces not

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Press, 1973); a long series of important articles by David Lewin leading up to *Generalized Musical Intervals...*, as well as other literature.

[7]Examples of this trend include Charles Wuorinen, *Simple Composition* (New York: Longman, 1979), a textbook in twelve-tone composition; John Rahn, *Basic Atonal Theory* (New York: Longman, 1980); Joseph N. Straus, *Introduction to Post-Tonal Theory* (Englewood Cliffs, NJ: Prentice Hall, 1990) and "A Primer for Atonal Set Theory," *College Music Symposium* 31 (1991): 1-26; Michael L. Friedmann, *Ear Training for Twentieth-Century Music* (New Haven and London: Yale University Press, 1990); Robert D. Morris, *Class Notes for Atonal Music Theory* (Hanover, NH: Frog Peak, 1991); and miscellaneous articles on pedagogy in *Journal of Music Theory Pedagogy*, *In Theory Only*, and other periodicals.

[8]Ibid.

yet composed. Of course, the analyst is also concerned with understanding and/or constructing the relations and categories among pieces in a given repertoire.

As an outcome of these differences, the composer's use of theoretic tools differs from the analyst's. The composer begins work with a general idea of the composition to be made. The idea will be eventually realized, if often altered, in the compositional process. At the onset, many different realizations of the idea will be possible; as work proceeds, some of these potential realizations will be dropped and others may present themselves. Thus the compositional flow can be modeled by a move from the basic idea to one of the potential realizations—a one-to-many relation. This kind of relation is not deterministic. In contrast, the analyst confronts completed compositions and studies both the unique, individual aspects of a work as well as its relation to other works of similar or contrasting types. A given piece may be included in a more general type or be cross-related to a different type. The analyst therefore proceeds from a diversity of pieces through types up to a repertoire as a whole. This process often involves many-to-one (inclusion) associations. Such associations are mappings and are deterministic.

Thus the pedagogy of compositional theory must be distinguished from that of analytic theory. Compositional theory must always respect the indeterminacy of compositional choice—choices that remain open until the composition is completed. Analytic theory addresses the way in which extant pieces are related in a repertoire, often to an ideal piece or an archetype. While this difference separates the teaching of "counterpoint" from "form and analysis," there has been a steady move from a compositional to an analytic orientation in theory pedagogy, especially in core courses. Many years ago, both basic and advanced theory were taught by composers. Students progressed by writing first exercises, then compositions in specific styles.[9] With the rise of professional music theory in the 1960s, non-composer theorists replaced composers, often teaching

[9]Because of this sequence, courses in theory were prerequisites for free or original composition.

students to analyze music rather than to compose it.[10] While written exercises are used at the beginning of study, later on the activities are analytic and critical. Even the role of counterpoint in the curriculum has changed; it is no longer there to provide a basis for composition but to study the underlying principles of voice leading as an adjunct to doing reductive analysis.[11]

These changes probably reflect the fact that music theory is presently taught to *all* students in a music department or school, most of whom are performance majors. As a result, theory is devoted primarily to developing critical musical skills that relate structure and analysis to musical performance. This move has often placed less emphasis on developing advanced skills in ear training, sight singing and keyboard. While the emphasis on analysis has provided a larger place for the study of twentieth-century music in the core theory curriculum, training of the ear, voice, and hand in comprehending atonal music is often omitted. As a result, many students are convinced that atonal music is completely cerebral and not based on practical and audible (read "natural") relations among pitches and pitch-classes. This attitude is exacerbated by the formal and mathematical appearance of atonal theory and the lack of importance it places on other salient features of twentieth-century music such as rhythm and timbre.

The last point about "other parameters" in twentieth-century music cannot be addressed at any great length here, but some response is appropriate. While it is true that what helps makes new music new is its exploration of all the possible dimensions of music, much of the fuss concerns not music theory—the study of musical structure—but musical style and repertoire. Atonal music theory needs to address how new conceptions of and roles for rhythm, texture, timbre, and spatial location help or hinder pitch and pitch-class relations and vice versa. For instance, in tonal music there is "tonal rhythm," but what about "nontonal rhythm?" Can there be

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[10]Consequently, young musicians have little understanding of what composition is about, resulting in a potentially unhealthy separation between composers and performers.

[11]In fact, counterpoint is often not in the core curriculum at all, except as an upper-division course.

general concepts of harmonic-rhythm, timbral-rhythm, x-rhythm (where x is any musical dimension)?[12] A good deal of research into the question of other parameters has yet to be done.[13] Until then, when we have moved beyond labeling to function, there can be no general pedagogy of all dimensions of musical structure, comparable to what we can impart about pitch/pc and, to a lesser extent, about rhythm in twentieth-century music. And there's plenty to teach about atonal structure, even if its relation to other musical features is unclear or solely contextual. The reader should not take my remarks as attempts to avoid the issue. In surveys and especially aural skills classes, considerable emphasis should be placed on appreciating and accurately identifying the music's rhythmic, articulative, textural, and timbral attributes.

2. *Entity versus Transformational Theories.*

The previous discussion talked about pieces as if they were primarily objects or entities. On the other hand, many conceptions of composition and analysis consider pieces as processes, often teleological in character. The former conception sees the piece as objectively "out there" in some kind of "reality," while the latter regards pieces as intersubjective experience, both useful points of view. The various dialects of atonal music theory can be divided somewhat similarly. Some theories consider musical entities primary—notes, time-points and their grouping into sets: ordered, partially ordered, and so on. Other theories consider the relation *between* entities as primary: the intervals between notes, durations between time-points, etc. The transformational approach suggests the generalization of

[12]Clearly, in twelve-tone music there is *aggregate-rhythm*, the velocity at which twelve-tone units turn over. Moreover, composer/theorists such as Messiaen, Hindemith, Schoenberg, and Babbitt, to name just a few, developed or invented ways in which musical time relates to and influences pitch and pitch-class structure.

[13]Contour theory seems to be a promising entrée into such matters; for an introduction to this field see Friedmann, "A Methodology for the Discussion of Contour: Its Application to Schoenberg's Music," *Journal of Music Theory* 29/2 (1985): 223-248; Marvin and Laprade, "Relating Musical Contours: Extensions of a Theory for Contour," *Journal of Music Theory* 31/2 (1987): 225-267.

intervals to transformations from one entity (or interval or transformation) to another. While a strict mapping of objective conceptions to entity theories and intersubjective conceptions to transformational theory is perhaps a bit too rigid, it is clear that this alignment of the two pairs is not counterintuitive. Nevertheless, a given piece may be addressed with either type of theory, although textural quantization is a determining factor.[14] Obviously, when the role of analysis is to appreciate how pieces go, then a transformational approach is relevant. An entity approach might be well suited for composition studies.[15]

3. *Formal and Informal Theories.*

In a discipline where courses called "music theory" are usually devoted to developing skill or technical know-how, assuming a "theory" that is never addressed as such, there is naturally some misunderstanding about what a theory is. When these confusions are joined with a more general misunderstanding and mistrust of science and mathematics, even the very term *theory* develops a number of negative connotations. Perhaps as an attempt to make theory more palatable, teachers often assert that some kinds of theories, deemed "informal," are more "user friendly" than others.

Actually, a formal theory has to show that all of its statements are truly derivable by logical rules from its basic, consistent assumptions, its axioms. The amount of detailed reasoning to do this even in relatively simple cases in mathematics is really beyond human ability; thus, most sophisticated and complex theories are actually informal, especially the so-called elegant ones, since they leave out many of the smaller logical steps in order to present the theory in

[14]Not considering how and how much a piece is broken into parts can lead to a dilemma. Even if each part of a piece A can be described as a transformation of another, A may be still be experienced as a series of states or events. And conversely, a piece B might be described as a series of states, but be heard as a continual flow from one state to the next.

[15]A theory that places entities in a transformational, especially a teleological, context may not stimulate original use of musical materials. For this reason, knowledge of one's materials independent of any preconceived actual or potential use can be an important adjunct to making original pieces.

the most succinct way for experts.[16] It is assumed that such omitted statements will be inferred and supplied by the professional.[17] Most so-called "formal" theories are in fact informal to one degree or another.

For a general audience (including musicians and music scholars) what is probably meant by "informal" is a theory without mathematics, or even better, without technical terms and concepts. Such theories can exist, but they soon collapse due to their own weight. Imagine how difficult it would be to explain adequately the notion of interval-class without the aid of terms such as unordered, set, pitch-class, mod-12, interval, absolute difference. Of course, one could point to and elaborate on symbols in a musical score or listen to and discuss certain sounds, but this presupposes an expertise in score reading and/or musical dictation determined by a working, if not definitive, knowledge of the terms just listed.

It is also commonly supposed that a theory is a closed field of discourse, as opposed to non-technical discourse whose openness is conducive to the imagination. Nevertheless, no theory is ever complete; there are always more questions that the theory can raise, some of which become celebrated problems in the field. It is also commonplace to find ways in which well-established parts of the theory can be altered so as to connect them with other theories or simply to make them more elegant and direct.

In any technical field—and given the years of training it takes to develop musical skill, music is such a field— some degree of formality is useful, indeed necessary to proceed. Clear definitions, symbology, and rules of inference make precision possible. Of course, one can easily go too far in this direction. After all, music is already technical with its own terms, concepts, and notations, even if these

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[16]Even experienced readers have difficulty in following the logic by which statements in a mathematical proof proceed. Sometimes the means of derivation are given simply by asserting that statement B follows from statement A "by algebra."

[17]This makes reading the advanced literature of a technical field almost impossible for the novice, no matter how strong his or her motivation and intelligence. What is obvious for the expert is a matter of lore, learned by practicing the field.

are not well defined. The problem is that, while the traditional terms are clear enough to the expert in tonal music theory, they are useful in atonal music only if they are considerably adapted, refined, and supplemented with newer concepts and notations. I address such issues in the next section. And while I understand that reducing the mathematical aspects of atonal music theory to the absolute minimum has its advantages, the precision and generality gained by using a little high-school math will be appreciated later, regardless of student opinions about "formality."

But can we do without the math? There is a trend to teach undergraduates atonal theory with the absolute minimum of formal means; pcsets become chords, set-classes become harmonies, and so forth. To this end a lore of specific strategies has been developed[18] over recent years; I applaud this development in as much as it reduces gratuitous pretension and any attendant intimidation.[19] Nevertheless, the attempt to "introduce" a subject without technical fanfare will not help later when the student studies it in detail. Shouldn't any introduction to a field of inquiry, especially at the college level, be just that, not merely a simplification? Certainly theorists who teach tonal music do not simplify or avoid concepts such as "neighbor chord," "exchange of voices," "prolongation," and the like when they introduce tonal theory to undergraduates. (And there was a time when such concepts were controversial or deemed too difficult to teach in the core.) Still it will be argued that the formal/mathematical aspects of atonal music theory are simply too much for the average undergraduate, especially a music student, to learn. Aside from the patronizing flavor of such views, the

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[18]For instance, set-classes are learned by association with particular pieces ([014] is the opening of Béla Bartók's *Music for Strings, Percussion and Celesta*; [026] is the beginning of the same composer's *Concerto for Orchestra*, fourth movement). Or to find the ic of a pitch-dyad, just reduce the size of the dyad by any number of octaves until the dyad is as small as possible and count the semitones.

[19]Joseph Straus's pedagogically oriented article "A Primer" probably goes as far as one can in reducing intimidating detail while preserving intelligible discourse. While the article is "adapted" from Straus's textbook, *Introduction to Post-Tonal Theory*, it succinctly describes the history and state of reseach in "set theory" and provides many useful citations to the technical literature.

most cursory examination of the textbooks for "average students" in other fields—in sociology, psychology, economics, foreign language, for instance—will make it plain that such disciplines are not loath to carefully define symbols and formulas, and demand that students follow complex chains of reasoning and memorize key terms and their applications as part of the *introduction* to their subject.[20] Even so, some will still object to "formalism" in music study as a necessary evil at best, and perhaps they are right in the largest scheme of things. For success in teaching is achieved when the student has gone far beyond cognitive competence and the subject has become "second nature." Such transparent mastery does not come easily; it is nurtured by clarity of presentation and purpose, supported by diligent practice and drill. In the end, there are no shortcuts.

Specific Recommendations

Up to now, I have been interested in differentiating types of theories and their applications. The following recommendations are designed to facilitate the teaching of atonal theory in composition classes and tutorials, and in analysis classes. I also consider them suitable for use in musicianship programs and workshops, and in undergraduate core music theory curricula. Exactly how these suggestions will be used depends on the instructor, the level of the students, the nature of the course, and the music speciality addressed. But no matter what the pedagogical intention, a balance of calisthenic exercises, readings and practice in analysis, and some composition and improvisation will help imbue the student with a healthy attitude toward atonal music and its literature.

[20]My own experience teaching elements of atonal theory to high school students (at the Governors' School of Pennsylvania, summers of 1976-9) and music undergraduates (at the University of Pittsburgh and The Philadelphia University of the Arts) has been most encouraging. Most of the students found the field challenging but not overwhelming; many of them enjoyed learning about pitch-class structure in new music.

Terminology and Notation

Several different notations for what are essentially the same concepts have arisen in the literature. Amidst this surfeit, two approaches are readily identified: those associated with work accomplished in the 1950s and 60s at Princeton and at Yale Universities. The Princeton contribution was slightly earlier, somewhat composer-oriented, and found its focus in the study of the twelve-tone system. The Yale "school" was primarily analytic in orientation, originally concerned with the pre-twelve-tone but post-tonal music of Schoenberg, Berg, and Webern and other contemporaneous composers such as Stravinsky.[21] There were also a few individuals working outside of these two factions, such as George Perle and Howard Hanson.

Although the field is much more homogeneous now, three strains of terminology have persisted: from Princeton, Yale, and beyond. Table 1 lists some of the correspondences in terminology among the Princeton, Yale and "other" orientations. As the reader can see, terms that are equivalent in mathematics and/or logic are differentiated ("set" versus "collection"); other terms are conflated ("set-class"); one basic term overlaps ("set"). With only a few exceptions, I take the view that students should be familiar with all generally used terms, even if they are redundant. This familiarity helps promote better communication among theorists working within different traditions of research. A certain degree of flexibility of terminology is also useful to prevent rigidity from creeping into discourse. Even so, it is better to teach using the terminology of one approach as primary so that the terms will be properly associated within the orientations of older research.

[21]While Rahn's *Basic Atonal Theory* of 1980 and Straus's *Introduction to Post-Tonal Theory* of 1990 present a somewhat hybrid approach to atonal theory, the two still reflect their respective origins in the Princeton and Yale orientations.

Table 1. Correspondences in terminology

Princeton	*Yale*	*"other"*
ordered interval	interval	
unordered interval	interval-class (ic)	
collection	set	pcset, cell
collection-class, set-type	set-class	
normal-form representative	prime-form	
set	row	series
set-class		row-class
symmetry	invariance	

Below I make various recommendations on terminology and notation.[22]

Names of pcs: The convention is to use integers (mod-12) for pcs. If 10 and 11 are used for A-sharp/B-flat and B-natural, lists of pcs need to be separated by commas to distinguish 10 (ten) from 1 0 (one, zero), for instance. To avoid the commas, letters for numerals have been proposed: ten is denoted by A, a, t, or T; and eleven is denoted by B, b, e, or E.[23] Although I introduced the A, B notation (in 1974 with Daniel Starr)[24], the t and e are easy to remember, and

[22]My suggestions that certain symbols, terms, and notations be employed in atonal theory pedagogy does *not* spill over into original research in the theory and analysis of twentieth-century music. Far be it for me to assert in advance how my colleagues should conduct their research!

[23]a and b are used only in certain computer programs. T and E are used solely by Straus in *Introduction*. X and Y have also been used as pc constants, but I discourage this since X and Y are so frequently used as variables.

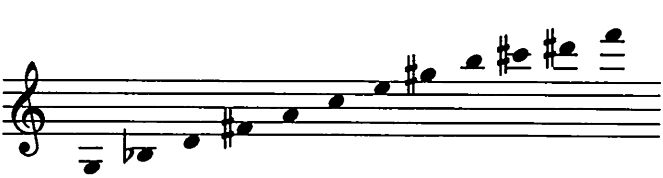
[24]See Morris and Starr, "The Structure of All-Interval Series," *Journal of Music Theory* 18/2 (1974): 364-388.

do not get confused with A and B as pitch-class letter names or with names of sets.

Some theorists have used letter names (such as A#, D♭, and the like) for pcs. This convention is used not only to reach out to those in other musical disciplines by avoiding a mathematical appearance, but also has a useful function in transformational theories. It indicates that the pcs are not the members of (or isomorphic to) a mathematical group, but are musical things that get transformed under musical transformations. While I am in great sympathy with transformational theories, the problems of enharmonic equivalence and the difficulty in reading longer strings of these symbols weighted against the facility of pc integers in more advanced contexts (such as the construction and use of *invariance matrices*) argue against letter names for pcs.

I generally discourage the use of staff notation for pcs, even though it is sometimes employed by major theorists and/or composers such as Babbitt and Wuorinen. Notes on a staff can easily lead students to confuse pcs with their realizations. For instance, I often begin the discussion of the pitch/pc distinction for composers with Example 1 written on the blackboard. I ask what is written on the board. The correct answer is not "the row" of Berg's *Violin Concerto,* but "a (pitch) realization" of Berg's row.[25]

Example 1. A realization of the row of Berg's *Violin Concerto*



[25]While on the subject of teaching the pitch/pc distinction, it is very important to show that every melodic presentation of a pcset is ordered in not just one, but in at least two dimensions, usually pitch and time. This observation leads directly into the discussion of musical contour, since a contour is the ordering of elements according to two or more dimensions.

Variables and constants: Capital letters, with or without subscripts or superscripts, are used to denote both sets and operations. Set and operator variables should avoid the following letters which are constants: A (a pc), B (a pc), I (inversion operator), M (multiplicative operator), O and P (classical row names), R (retrograde operator), S (classical row name or set), T (transposition operator), e (a pc), t (a pc). Be aware that: r (with subscript) has been used as the rotation operator; E (with subscripts) is often used as the name of an array or matrix; F, G, H, K are often used as operator variables.

Set and operator (transformation) names: These should be notationally differentiated. The standard is left autography: the notation of a set and its transformations is written so the name of the set is on the right and the series of transformations on the set is listed from right to left.[26] For example, T_5IRX indicates that the set X is first retrograded, then inverted, then transposed by 5. The reasons for adhering to this convention are many. For instance, while a subscript n on a set name locates the nth element of the set[27], a subscript on a operator name designates specifics of the operation. Thus we avoid the standard but unfortunate row labels such as P_5,[28] R_4,[29] I_7,[30] RI_3 to indicate what is properly notated as T_5P, RT_4P, T_7IP, RT_3IP.[31]

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[26]In general usage, T_n stands for pitch-class transposition and T_nI for pc inversion followed by pc transposition. In specific cases, the subscript in the transposition operator stands for the interval of transformation (as in T_3, for instance). Sometimes, in more formal discourse, the subscript can be a mathematical expression (as in $T_{(5+b)}$ or T_{-n}, where b and n are variables).

[27]Elements of a set can be notated with subscripts starting from 0 or from 1; the convention employed should be announced up front.

[28]P_5 would stand for the fifth element of P.

[29]The R operation doesn't take a subscript. In addition R_4 is ambiguous. In the literature, R_n might mean either the retrograde of T_nP or the transposition of RP that begins on pc n.

[30]I_7 has been used as an operator label equivalent to our T_7I. This has merit in more advanced contexts where I_7 is a member of the mathematical group of pc operators since, in general with the notation T_nI, a single operator in the group has to be notated as a concatenation of two operators. On the other hand, there are analytic contexts where it is preferable to use IT_n rather than its equivalent, $T_{-n}I$; operators of the form I_n will not allow this option.

[31]The separation of the operator from the name of the row P also allows us to identify more than one row-class (a row and its 48 transforms)

Set-theoretic Notations: The following concepts and notations should be introduced at an early stage.

$\cup$ union	$\cap$ intersection
$\subset$ set inclusion	$\in$ membership

′ complementation (i.e. A′ is the complement of A)

Brackets: Another useful convention involves the use of different kinds of brackets for different functions.

{ and } surround unordered sets.
< and > surround ordered sets.
[and] are used in special cases such as interval-vectors or set-class names.
(and) are used in mathematical (set-theoretic) expressions and functions.

Brackets can help show certain kinds of pc constructs.
A partition of the aggregate into major or minor chords:

{ {047} {259} {6t1} {8e3} }

The set-class 4-9[0167]:

{ {0167} {1278} {2389} {349t} {45te} {056e} }

Partially ordered sets (posets) can be also be notated readily. Below we have the poset Z consisting of an ordered set starting with an unordered set (pcs 0 3 and 5), followed by an unordered set comprised of two ordered sets (5 followed by 4; 2 followed by 1), finishing up with the ordered set t e.

in a given analytic context. So if a piece uses, say, two independent twelve-tone rows, we can refer to these as P and Q and distinguish RT_5P from RT_5Q.

Z = < {035} { < 54> <21> } < te> >

Some sequences of pcs that satisfy Z follow:

3055214te   5035421te   0532514te   0355241te

Set-classes: The concept of a set-class (set-type, or collection class) can take at least three forms. A set-class is: (1) the set of all pcsets that are related by T_n and/or I; (2) a type (of pcset) having tokens related by T_n and/or I; (3) any member of a group of sets related by T_n and/or I. The first interpretation—a set-class is a set of pcsets—is preferred for the following reasons. First, the normal-form representative (prime-form) is shown to be only an arbitrary member of the set-class, fixed by an algorithm which, operating on any pcset in the set-class, returns the representative.[32] Second, the nominal nature of the definition avoids considering the set-class as a Platonic object. Third, such confusions as T_5(3-5) (or alternatively T_5[016]) are avoided. Fourth, the definitions of abstract inclusion and complementation are easily defined. Fifth, the definition implies that hearing set-classes is not a matter of identifying a particular collection of pitch-classes, but of identifying intervals (and their configurations) in the realizations of pcsets in pitch and time.[33]

A clear distinction between pcset and set-class allows one to separate pcset inclusion and complementation from set-class inclusion and complementation. I have called the former *literal* and the latter *abstract*. I provide the definitions below:[34]

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[32]One typical problem one often sees in the work of novices is exemplified by the statement "The pcset {567} found in m.5 is the T_5 of the prime-form of set-class 3-1[012]." It is irrelevant to note such a relation, especially when the pcset {012} never appears in the composition analyzed.

[33]In class, I encourage locutions such as "{129} in m.5 is a member of set-class 3-4[015]," as opposed to "set-class 3-4[015] occurs in m.5."

[34]Note that these definitions apply for all kinds of set-classes (see below).

Let X and Y be pcsets. XX and YY are set-classes. U is the set of all pcs, the aggregate, and {} is the empty set.

(1) X is *literally included* in Y if every pc in X is also in Y ($X \subset Y$).

(2) XX is *abstractly included* in YY if $X \in XX$, $Y \in YY$, and $X \subset Y$.

(3) X is the *literal complement* of Y if $U = X \cup Y$ and $\{\} = X \cap Y$.

(4) XX is the *abstract complement* of YY if $X \in XX$, $Y \in YY$, and $Y = X'$.

Literal and abstract relations can be extended to similarity relations.

The name of a set-class takes two forms. The Rahn (Princeton) name is a set-class's normal-form representative, while the Forte (Yale) name is more arbitrary, derived from the position of the set-class on Forte's list.[35] I strongly suggest theorists use both names at once in identifying set-classes. For instance, rather than using [012367] or 6-5 alone, use 6-5[012367]. This will allow theorists from both the Princeton and Yale approaches to follow the theoretical argument.

Each of the two styles of set-class names has its advantages. The Rahn name provides the user with an example of a member of the set-class. This is very useful in the early stages of pedagogy and for this reason Rahn's names for the twelve trichordal set-classes probably should be the standard. Forte's names have different advantages. The name of a set-class is coordinated with the name of its abstract complement; thus the set-class 5-3 is the abstract complement of the set-class 7-3.[36] The Rahn names for these set-classes,

[35]See Appendix 1, "Prime Forms and Vectors of Pitch-Class Sets," in Forte, *The Structure*, and Table II, "T_n/T_nI-Types of Sets," in Rahn, *Basic Atonal Theory*. Similar tables are available in Morris, *Composition* (Appendix One, 313-19) and *Class Notes* (Appendix 1: "Set-Class Table," 105-111) and in Straus, *Introduction* (Appendix 1 "List of Set Classes," 180-183). The latter reference has another set-class table (Appendix 2 "Simplified Set List," 184-214), which lists all pcsets from three to nine pcs in ascending order with their set-class affiliations. My worries about the use of computer software to look up the set-class of a pcset also apply to this list (see note 56).

[36]Unfortunately, since Z-related hexachords are mutually abstract complements, Forte's hexachordal names do not indicate abstract complementation. For instance, 6-6 and 6-38 are abstract complements.

[01245] and [0123458], do not easily indicate their abstract complementarity. And while Rahn's labels are useful for set-classes whose members have few pcs, they become cumbersome with set-classes containing pcsets with six or more pcs. For instance, the Rahn names [013468] and [013568] might easily be confused, where the corresponding Forte names, 6-24 and 6-25, are easy to distinguish.[37]

I also prefer to use Martino's labels, the letters A through F, to designate the six all-combinatorial hexachords; thus,

A = 6-1[012345]
B = 6-8[023457]
C = 6-32[024579]
D = 6-7[012678]
E = 6-20[014589]
F = 6-35[02468t]

While set-classes are based on T_n and I relations, other kinds of set-classes can be constructed for certain purposes and occasions.[38] The canonical group[39] need not be only the set of T_n and T_nI operations; it can be T_n alone, or T_n, I and M together, or other, non-distance-preserving operations such as Mead's O_z,[40] or even T_0 and T_6 alone.

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[37]Some theorists have objected that one has to memorize Forte's table of set-classes just to make use of his names. It is true that Forte's names provide no information to help identify the members of a set-class. On the other hand, his short names are easily communicated. The same reasoning explains the common use of Martino's designations A-F for the six all-combinatorial hexachordal set-classes.

[38]Hanson in *Harmonic Materials* and Forte in his article "A Theory of Set-Complexes" proposed the following definition of set-class: two pcsets A and B are in the same class if the interval vector of A is the same as the interval vector of B. This collects Z-related set-classes into one class. The intuitive appeal of this kind of equivalence is counterbalanced by certain problems that arise associated with abstract inclusion.

[39]*Canonical group* is Lewin's term for the mathematical group of operators that defines the relation among members of a set-class. See Lewin, *Generalized Musical Intervals*,104.

[40]See Andrew Mead, "Some Implications of the Pitch Class/Order Number Isomorphism Inherent in the Twelve-Tone System: Part One," *Perspectives of New Music* 26/2 (1988): 96-163.

Invariance: This term indicates the case where, given a pcset X and an operation H,

$$X = HX.$$

Rahn's function,[41] which I call *degree of invariance,* borrowed from group theory, is a valuable tool.[42] The degree of invariance of a pcset Y, written D_Y, is the number of operations that keep Y invariant. The number of pcsets in a set-class including pcset Y is D_Y divided by the number of operations in the canonical group.

I usually avoid the noun *invariant,* presumably meaning a pc that does not change under a particular operation, or the locution "operation so-and-so holds the pcs x, y, etc. invariant." While there are pcs that do not change under some T_nI operations (such as pcs 0 or 6 under T_0I), "invariant" is too often used to refer to pcs as in the following statement: "under T_6, pcs 1, 4, 7, and t are the invariants." Now, while it is true that the pc*set* {147t} is invariant under T_6, the individual pcs *do* change: 1 becomes 7, 4 becomes t, etc. So, strictly speaking, there are no "invariants." And the locution can be easily fixed: rather than saying "T_6 holds the pcs 1, 4, 7, t invariant," say "T_6 holds the pcset {147t} invariant."[43] Perhaps all of this sounds a bit pedantic, but if students are to go on to study the cyclic representation of operators, cyclic adjacencies, and/or Lewin's injection function, attention to these distinctions will help pave the way.

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[41]He calls the function *degree of symmetry.* See Rahn, *Basic Atonal Theory,* 90. I usually avoid the term "symmetry," although it is roughly equivalent to invariance in some group-theoretic contexts. One reason is that symmetry often implies mirror symmetry in pitch and there is a class of pcsets that cannot be realized in pitch with a one-to-one correspondence from pitch-class to pitch.

[42]For example let Y be {02468t}. Y is a member of the whole-tone scale, invariant under T_0, T_2, T_4, T_6, T_8, T_t, T_0I, T_2I, T_4I, T_6I, T_8I, T_tI. Since there are twelve operations that keep Y invariant, $D_Y = 12$.

[43]Another instance of this misconception occurs in twelve-tone theory, where students often talk about the "invariant pcsets" in a row. For instance, given P = <02354768t1e9> and T_5IP = <53201te97468>, the sets {0235}, {4678} and {19te} are called "the row's invariant sets," "row invariants," or some such because the pcsets are found contiguously in both rows. Rather, only {0235} is invariant under T_5I, the other two sets mapping onto each other under the operation. This explains why and how the relative positions of the three pcsets in P permute under T_5I.

Similarity Relations: These are various methods of comparing aspects of the sonic similitude between pcsets, usually of different set-classes. Beyond the following remarks, I refer the reader to articles by Rahn and Isaacson for material on the published similarity relations and their roles in composition and analysis.[44] The basic problems with the more refined measures of pcset similarity is that, even for just two pcsets, a fair degree of calculation is necessary to establish a specified degree of similarity. On the other hand, the simpler methods of determining similarity with a minimum of hand calculation have certain methodological problems. Nevertheless, the practicality of these easy methods makes them somewhat useful for at least pilot studies.[45]

When teaching similarity, the distinction between equivalence-relations (underlying the definition of set-class) and similarity-relations should be made clear.[46] Working out similarity networks among various set-classes provides interesting and useful projects.

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[44]See Rahn, "Relating Sets," *Perspectives of New Music* 18/2 (1980): 483-502 and Eric J. Isaacson, "Similarity of Interval-Class Content between Pitch-Class Sets: The IcVSIM Relation," *Journal of Music Theory* 34/1 (1990): 1-28 for reviews of other similarity measures. (I must point out two problems in Isaacson's otherwise useful paper: (1) My ASIM measure is SIM divided by the number of ics (not pcs) in the two pcsets compared (p.5); (2) The measures of Rahn and Lewin are properly performed using total abstract subset content rather than with only the ics of the pcsets compared (pp.8-13).)

[45]I am thinking of Forte's R_n and/or my SIM measures here. (See Forte, *Structure*, 46-60 and Morris, "A Similarity Index for Pitch-Class Sets," *Perspectives of New Music* 18/2 (1980): 445-60.) A combination of both pc and ic intersection as a criterion of similarity provides a decent gauge of pcset/set-class similarity. (Such a tack is taken in Forte's *Structure* in Appendix 2 "Similarity Relations," 182-99; he assigns similarity on the combination of his Rp with his R_1, R_2, and/or R_0 measures.) In any case, some similarity measures for ordered sets of pcs are given in Morris *Composition*.

[46]Equivalence and similarity relations are symmetric and reflexive, but only the former is transitive; in other words, X is not necessarily similar to Z even if X is similar to Y and Y is similar to Z. One could also introduce partial orderings as a third class of relation (reflexive, antisymmetric, and transitive). Partial ordering relations bridge the gap from (totally) ordered to unordered sets and model set/set-class inclusion and twelve-tone ordering among combinations of rows. Incidentally, genealogical relations make good examples for teaching these concepts. For instance

Notation of Ordered Sets and Order Relations: Although pcsegs are easily notated, in serial theory certain abstract relationships between and among rows and other ordered sets become awkward. For instance, if we wish to talk about the pcset formed by the pcs at order numbers 3, 5, and 7 in the row W we must write $\{W_3\ W_5\ W_7\}$—precise but awkward. Andrew Mead has defined a few conventions that are helpful in such cases.[47] Given a serial entity, S, we can identify the pcs at order numbers in S by listing the order numbers alone and writing them in boldface (or in written work, with underlines). Our example with row W would be written {**357**}. Order operations that are isomorphic to pc operations are noted in boldface as well so that, for instance, $\mathbf{T}_n$W denotes order number transposition, the rotation of W starting with the pc in W's (12–n)th order position and wrapping around until we end with the pc in W's (12–n–1)th order position.[48] **TeI** is equivalent to the R (retrograde) operation, and so forth.

In general, Mead begins with Milton Babbitt's original definition of a row: a set of twelve couples (ordered sets) of discrete order numbers and pitch numbers.[49] The row of the Berg Violin Concerto would be written:

$$\{ \langle 0,0\rangle \langle 1,3\rangle \langle 2,7\rangle \langle 3,e\rangle \langle 4,2\rangle \langle 5,5\rangle \langle 6,9\rangle \langle 7,1\rangle \langle 8,4\rangle \langle 9,6\rangle \langle t,8\rangle \langle e,t\rangle \}$$

Mead then defines two aspects of a row: a *pitch-row* and an *order-number row.* A pitch-row is the traditional way of writing a row: the pcs are written according to their ascending order numbers; an order-number row writes the order numbers according to their pcs in ascending order. As an example, the preceding row is written in both aspects.

"Father of" is a partial ordering; while "In the same family of" is an equivalence relation, "Sibling of" is not, since it is symmetric and transitive but not reflexive, and so forth.

47See Andrew Mead, "Some Implications...:Part One."

48$\mathbf{T}_n$ is equivalent to the rotation operator r_n.

49While it is an historical accident that Schoenberg called his "Reihe" a "set," Babbitt's formalization *is* a set, since his definition as given above is a set (unordered) of ordered couples.

Pitch row: <037e2591468t>

Order-number row: **<07418592t6e3>**

Mead's notations are extremely useful in studying partitions of pc or order numbers. For instance, the following order-number partition applied to Berg's violin concerto row yields a corresponding pc partition and vice versa; this order-number partition is used at the opening of the piece (G-natural is pc 0).

{ **{0246} {1357} {89te}** } ↔ { {0729} {3e51} {468t} }

Hearing and identifying pcsets and set-classes

I now turn to the second part of this paper, a subject of utmost importance if atonal theory is going to have any real impact on the musical habits and sensibilities of its hearers. Certainly one can get used to the "sound" of atonal music and really develop a taste for it; the question is whether the relations among its pitches and pitch-classes can have a degree of correspondence comparable to the function that pitches and pitch-class relations hold in tonal music. If not, then perhaps atonal theory is only for specialists. Judging from the statements of composers committed to and experienced in atonal music composition, pitch and pitch-class relations are both hearable and essential to the understanding and appreciation of this music, in ways that resemble and/or contrast with older music. Based on my own teaching experience, students with good, but not outstanding musical ears, can learn to hear and respond to things atonal so as to follow an atonal composition's progress by ear to some extent.[50] But I am also concerned with the development and mastery of advanced hearing skills that are needed for those student composers, theorists, musicologists, and musicians who might special-

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[50]Of course, listeners to any kind of music aurally pick up much more than what they can articulate with some sort of technical musical discourse. I believe that teaching musicians to be able to identify and name what they have already perceived but cannot yet discuss expands the scope and depth of what they can perceive.

ize in twentieth-century music. I am further convinced that those students who have evidenced superior aural skills in tonal music can achieve equivalent degrees of expertise with atonal music.[51] To this end, I will provide an outline of various grades of difficulty of aural and visual identification of set-classes.

Much of what follows is certainly influenced by Michael Friedmann's excellent textbook, *Ear Training for Twentieth-Century Music,* cited in note 7. I recommend this book highly to anyone who wishes to teach atonal ear training to undergraduates. I should also say that my present concern (identifying set-classes and transformations) is only a subset of what Friedmann sets out to accomplish and that my methods sometimes differ from his. Although I shall identify the most important differences as I proceed, I consider my recommendations to be supplements to, rather than replacements for, his ideas.

Perhaps the greatest stumbling block in learning to hear set-classes[52] derives from the way students are usually taught to identify to which set-class a given pcset belongs. For instance, when presented with the pcs {50t291}, the task is to determine its set-class.[53] There are two basic ways: use a "normal-form" algorithm,[54] or derive the interval-vector of the pcset and look it up on a table.[55] While the second method appeals to me more (since it involves profiling the interval-class spectrum that the pcset shares with other members of its set-class), with larger set-classes both methods involve

[51]Nor do I believe that absolute pitch is a necessity in advanced atonal ear training. Since atonal pitch and pc relations are based on ordered and unordered intervals, a good sense of relative pitch is sufficient.

[52]Of course one cannot "hear" a set-class. A set-class is a set of pcsets. It should be clear that the skill I am after involves determining to which set-class a heard pcset belongs. One assumes that it is possible to tell that two heard pcsets are or are not members of the same set-class without naming the class.

[53]The answer is 6-14[013458].

[54]Using Forte labels alone, the normal-form (actually "best normal-order") transposed to begin on pc 0, the "prime form," is not the set-class name. One must look the prime form up on Forte's table to get the Forte name.

[55]This method is not altogether direct since the pcset might be a member of a Z-related set, so one has to choose between two entries on the table.

calculations that most of us find difficult to do in our heads. Such methods are better used in computer programs. Indeed, I rarely use either method.[56]

What other musically relevant ways are there to identify set-classes quickly? The answer is probably obvious; learn first to identify the six interval-classes (the set-classes whose members are pcsets of two pcs), then go on to memorize the trichords. Set-classes with pcsets including more than three pcs can then be identified by their abstract ic and trichord content. Two problems arise. (1) How to organize the move from smaller to larger pcsets and their set-classes? (2) How to identify a set-class in all of its manifold pitch and temporal realizations and environments? I shall take up (2) first.

The basic problem is hearing and identifying interval-classes in all manner of musical passages. Fortunately, the pedagogy of tonal ear training and sight singing provides some experience in identifying and producing intervals. Nevertheless, much time and energy should be allocated to aural identification of both ordered and unordered intervals as presented as pitch pairs (in time). Exercises[57] such as those in Example 2 will prove useful.[58] These can be done as aural or written exercises. Since timbre is so important in twentieth-century music, the instructor/auditor might aurally present the

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[56]There are of course a number of useful programming packages for automating various atonal music tasks such as the CMAP utilities by Alexander Brinkman and Craig Harris. And it is relatively easy to write a simple program that will identify the set-class of a pcset. Nevertheless, too early a reliance on such programs will indefinitely delay the internalization of set-class identification, contributing to a lack of confidence in the value of atonal pitch/pc relations in music.

[57]While the examples or exercises I have provided were composed for this occasion (primarily to avoid a plethora of permission notices), most of the music for these tasks can be culled from the literature. As I comment later, teachers of atonal theory need anthologies of twentieth-century music written by major composers selected and balanced for pedagogical goals.

[58]The reader will note that some of the examples have more than two pitches. Exercises involving pcsets of more than two pcs should also involve octaves and unison "doublings." Simultaneous pitch-class duplication is often employed even by composers such as Schoenberg and Webern despite prohibitions against octaves found in their writings and lectures.

exercises with a multi-timbral synthesizer ("keyboard") perhaps using sampled instrumental sounds. The student should be asked to sing and play (on any instrument) intervals and ics, at first freely, then with the instructor providing the first, last, top or bottom pitch of the ic to be realized.[59] Once mastery in ic identification is secured, one can go on to larger pcsets. Examination of their interval (class) vectors will become meaningful to the student who can hear ics.

Returning to problem (1) above, I will plot a series of five stages leading to the identification of *all* the hexachordal set-classes. The first two stages are basic, the next two intermediate, and the last advanced.

Stage 1. The identification of pc intervals and ics as just described.

Stage 2. The aural and written identification of the twelve trichordal set-classes.

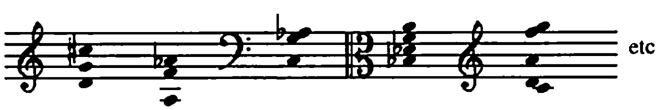
> Command of this stage provides the basis for competency with atonal materials. While the ability to detect ics in different pitch and temporal realizations will be of great help, the vast number of realizations of the pcsets from one set-class needs to be addressed. One might begin with trichords confined to small pitch ranges, then expand in width gradually to two octaves, a width which will accommodate the vertical ordering of the pcs in any trichordal pcset.
>
> To those students who have taken courses in tonal theory, several trichordal set-classes will be familiar: 3-10[036], the diminished triads; 3-11[037], the set of major and minor chords; and 3-12[048], the augmented triads. In addition, the chromatic trichord, 3-1[012] and the trichord 3-6[024], the first three notes of any major scale, will be easy to remember. Of course,

[59]Friedmann's Exercises 2.1 through 2.17 extensively drill visual and aural recognition of interval-class (called by him and Rahn, unordered pitch class interval), as well as the range of realizations of interval-class in pitch space (pp.9-21). My Examples 2a and 2d are in the style of his Exercises 2.12 and 2.17, respectively.

Example 2. Exercises in identification of interval classes

2a. Identify the ics. (Each example is played once.)

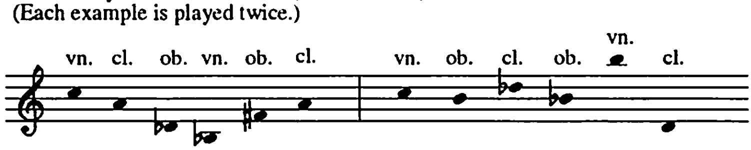
2b. Identify the ics between the top (bottom) two notes of each chord.
(Each example is played 3 times.)



2c. Identify the last (first) ic of each melodic sequence.
(Each example is played once.)

2d. Write down the succession of ics in the melodic sequences.
(Each example is played 3 times.)

2e. Identify the ic in the violin (oboe, clarinet)
(Each example is played twice.)



"unorthodox" pitch spacings, doublings, and orderings of the pcset members of these set-classes will still present problems.

One way to further differentiate the trichords and their pitch and time realizations is to divide them into the nineteen T_n-type set-classes. 3-1[012], 3-6[024], 3-9[027], 3-10[036] and 3-12[048] remain the same, but the other seven trichordal set-classes divide into a pair of T_n-type set-classes each containing pcsets related only by transposition; thus, 3-2[013] divides into 3-2a[013] and 3-2b[023] and so forth. (3-11 divides into 3-11a[037] containing only minor chords and 3-11b[047] containing only major chords.)

Yet another way to differentiate trichords involves the construction of what we may call *twelve-tone figured bass*. For our present purpose, we define a *fb-class* of pitch-sets. Each trichordal fb-class includes sets of (at least) three pitches. Each pitch-set in a fb-class is defined by (at least) two intervals: the ordered pc-interval from the lowest pitch to the second lowest; and the ordered pc-interval from the lowest to the highest. All trichordal pitch-sets in a fb-class have the same two pc-intervals "above the bass."[60] The name of the class is the listing of the two defining intervals (in order);[61] thus, 45 is the name of the fb-class that has two pitches at any pitch realizations of pc-intervals 4 and 5 "above" the pc of lowest pitch (or bass). Some members of 45 are given in Example 3.[62]

It is easy to show that the pcsets in any trichordal set-class may be realized as m distinct fb-classes. In the following

[60]Once the pc of the lowest pitch (the bass) is given, the pcs of the other two pitches have been determined.

[61]The name of one fb-class associated with the realizations of the pcsets in a set-class is the Rahn name of the set-class without the initial zero. For instance, 12 is a fb-class that can be realized by a pcset member of 3-1[012]. Of course, a zero in the fb-class indicates an obligatory (octave) doubling of the bass.

[62]Friedmann further differentiates trichordal realizations in pitch with his *rip* (registral pitch interval series) construct. A rip of a trichord is the set of its pitch realizations that are transpositionally equivalent in pitch. The underlying concept is equivalent to a pitch T_n-type set-class. Rip is the same as Morris's SP (spacing) of a pitch set. See *Composition*, 54-5.

Example 3. Members of the fb-class 45

formula for m, Y is any member of the set-class and D_Y is the degree of invariance of Y.[63]

$$m = 6 / D_Y$$

From the formula, seven of the trichordal set-classes can be realized as members of six fb-classes. For instance, 3-11 can be realized as the fb-classes: 37 49 58 47 38 59. These correspond to the traditional 53, 63, and 64 figured basses of minor and major chords respectively. Pcsets that have (only) T_nI-invariance have a degree of invariance of 2 so that there are only three fb-classes associated with them. This involves four trichordal set-classes. For example, the three fb-classes associated with 3-6[024] are 24, 2t, and 8t. Finally, 3-12[048] has a degree of invariance of 6, involving both T_n- and T_nI-invariance; thus it has only one fb-class, 48.

Aural and written drill can involve the fb-classes associated with a trichordal (or other) set-class. A few simple exercises are shown in Example 4.

Still another strategy for learning the trichords involves contrasting highly different set-classes and working up to more similar ones. For instance, one could start with 3-1[012] and 3-11[037] and then add 3-6[024]. A good set of contrasting set-classes (of pcsets of any cardinality) is the maxpoint sets.[64] For trichords these are: 3-1[012] maximizing ic 1, 3-6[024]

[63]The general formula for any set-class is 2 * #Y / D_Y. (Two times the cardinality of Y divided by the degree of invariance of Y.)

[64]A maxpoint set (pcset) is a set that maximizes a given interval class for its cardinality. See Tore Eriksson, "The IC Max Point Structure, MM Vectors and Regions," *Journal of Music Theory* 30/1 (1986): 95-111.

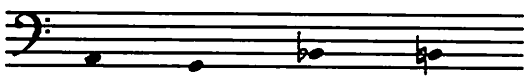
Example 4. Exercises in fb-classes

4a. Identify the fb-class of the following chords.

etc

answers: 48, 01, 36, 67, 1234e

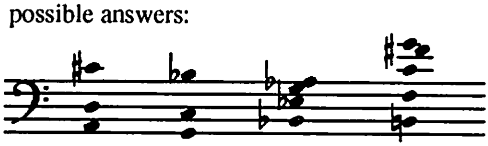
4b. Realize the fb-classes above the given notes.



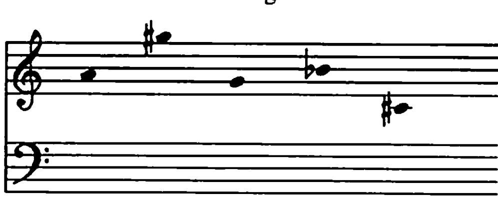
etc

45 35 59t 1678

possible answers:



4c. "Harmonize" the following
notes with the fb-classes given.



etc

68 47 68 47 14

possible answers:

maximizing ic 2, 3-10[036] maximizing ic 3 and ic 6, 3-12[048] maximizing ic 4, and 3-9[027] maximizing ic 5.[65] Incidentally, these five set-classes are the ones whose pcsets have T_nI-invariance.[66] In essence, the student will have to learn to identify the trichords without relying on calculation, on their "sound" alone. Cognitive aids such as listening for the presence or absence of particular ics can help at initial stages, but in the end, the response should be as automatic as identifying ics.

Stage 3: Identifying relations among ics or trichords.

This stage marks the beginning of intermediate mastery of recognizing some of the more important features of pitch-class relations in atonal music. By now, pieces by Webern, Babbitt and others that employ relations among clearly segmented trichords will become aurally lucid.[67] The student will be able to begin to hear how pitch and pitch-class function on both local and formal levels. In general, the student will have learned that each set-class (and its incarnations in T_n-type set-classes and fb-classes) has its own set of sonorous signatures. Audition will gradually involve much less effort and pieces that might immediately seem messy or featureless will come into focus. The student will begin to appreciate the cogency of good analyses of new music. And the new-found discrimination will help refine tonal hearing as well. With the more gifted students, the achievements gained in the first two stages may stimulate a desire for further development in atonal audition and identification.

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[65]Of the three set-classes whose pcsets maximize ic 6, I choose 3-10 (which also maximizes ic 3). Note that there is no pcset that has more ic 6s than any other other ic except for the interval of a tritone (ic 6).

[66]Friedmann emphasizes recognition of the T_nI-invariant trichords (called by him inversionally symmetrical) in his Exercises 4.15 and 4.16 (pp. 53-54).

[67]I am thinking of pieces such as Webern's *Concerto for Nine Instruments*, Opus 24, or his *Piano Variations*, Opus 27 (first movement), Babbitt's song cycle *Du*, his Second String Quartet, or his piano piece *Partitions*.

Stage 3 develops the ability to identify precise relationships among atonal music entities. For instance, the student is asked to specify how members of the same or different set-classes are related. Exercises may start off with comparisons of (short) unordered or ordered sets of pcs (pcsets or pcsegs), first realized as pitch sets related by pitch transposition and inversion. First the student is asked to tell only if the sets are related by T_n or T_nI; the identifications first compare ordered pcsets (pcsegs), then unordered pcsets. Exercises such as those found in Example 5 can help develop these skills.[68]

The next degree of difficulty is to identify under which specific operation a first pcset relates to a second. Keeping the pc sets related under pitch transposition, the student will be asked to identify the n in the T_n operation by identifying the interval between the two sets. With pcsets related by T_nI the problem is more difficult, not only aurally, but conceptually as well. In order to explain these difficulties, I must digress for a few paragraphs.

The conceptual cramps in pitch-class inversion are due to two factors. (1) Pitch-class inversion does not rely on contour—mirror inversion—in its realization in pitch. For example, consider the two chords in Example 6a. While the pcs of the first are related to the pcs of the second by T_3I, the pitch spacing of the chords disguises this relation. (2) Relation under a given T_nI operation does not imply a unique center-pitch. In Example 6b1, we have two pcsegments related by T_4I realized so that the resultant pitch sequences are also related by mirror inversion in pitch. The center pitch is D-natural. In Example 6b2, the same two pcsegments are used, but the upper one is realized an octave higher. Now the center of inversion is A-flat. Since the difference between the two examples is only in pitch, it is clear that the center pitch of inversion is not uniquely specified by T_4I. In fact, any two pc entities related by T_nI where n is even can be made to mirror around a center whose pitch-class is n/2 or (n/2)+6.

[68]In Exercises 4.7 and 4.27, Friedmann drills relations between multiple instances of pcsets.

Example 5. Exercises in comparing pcsets

How are the two note configurations related: under transposition or inversion or neither? (Each example is played twice)

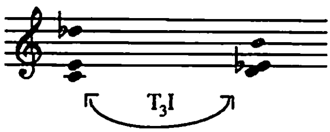
Example 6c1 and 6c2 show the same thing but under T_5I. Here the lower pcsegment and its pitch realization is the same as before, but the upper pcsegment is the T_1 of the former. Example 6c also makes the point that the center of inversion for T_5I has to be a quarter-tone. Since quarter-tones are not defined as pcs, there is no "center" of inversion from a pitch-class point of view, if by center we mean a single pitch.[69] In sum, this example shows that trying to teach, much less understand pc inversion, from a contour or center of inversion point of view is incoherent.

In my view, the best antidote to these problems is to specify pc inversion by its index number, the n in T_nI. This gives us twelve different inversion operators: T_0I, T_1I, T_2I, etc. The operation of inversion is then studied by considering what happens to the twelve pcs under each of these operations. For instance, under T_4I: pc 0 goes to pc 4, pc 1 goes to 3, pc 2 goes to itself, pc 3 goes to 1, pc 4 goes to 0, pc 5 goes to pc e, etc. We can write these associations as follows, first on pcs with mapping arrows, then as pcs related by double-headed arrows, and finally as cycles.

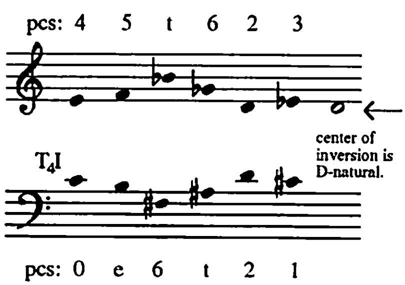
[69]We can say that the pcs related by T_5I have a pc center of the pc pair <2,3>, but <0,5>, <6,e>, and three other pairs will serve just as well.

Example 6. Pitch-class inversion·

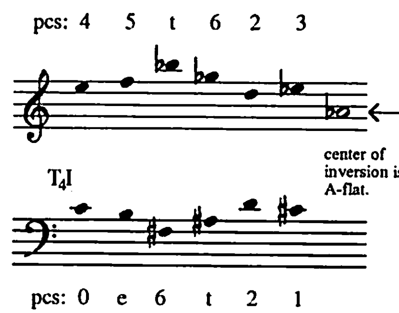
6a.



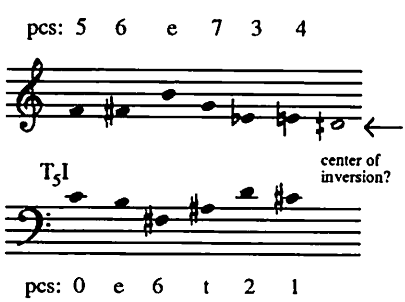
6b1.



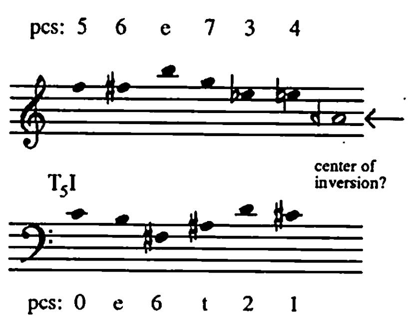
6b2.



6c1.



6c2.



$0 \rightarrow 4 \quad 1 \rightarrow 3 \quad 2 \rightarrow 2 \quad 3 \rightarrow 1 \quad 4 \rightarrow 0 \quad 5 \rightarrow e$
$6 \rightarrow t \quad 7 \rightarrow 9 \quad 8 \rightarrow 8 \quad 9 \rightarrow 7 \quad t \rightarrow 6 \quad e \rightarrow 5$

$0 \leftrightarrow 4 \quad 1 \leftrightarrow 3 \quad 2 \leftrightarrow 2 \quad 5 \leftrightarrow e \quad 6 \leftrightarrow t \quad 7 \leftrightarrow 9 \quad 8 \leftrightarrow 8$

(0,4) (1,3) (2) (5, e) (6,t) (7,9) (8)

The cycles of the T_4I operation tell us what happens to a pc under the operation: just find the pc in one of the cycles and look to its right, wrapping around to the first pc in the cycle, if necessary. These cycles are the verticalities of note-against-note inversion under T_4I as in example 6b1. Note that all the cycles if considered as intervals are even, ics 0 2 4 6.

Now let's look at the cycles of T_5I:

(0,5) (1,4) (2,3) (6,e) (7,t) (8,9)

Here there are only six, not seven cycles, all of them having two pcs. If a passage has note-against-note inversion under T_5I, the cycles will form the verticalities, just as before; but the sound will be different, for all the intervals are odd, ics 1 3 5. (See examples 6c1 or 6c2 for verification.) I could go on to rehearse other well-known facts: that T_nI inversion comes in two flavors, odd and even; that the pcs in the cycles add up to the index number; and other important properties. Suffice it to say that once the students are aware of the cycles (intervals) that underlie pitch-class inversion, they will have something reliable and definite to identify and hear.

Identifying pitch-class inversion can be addressed by exercises that ask the student to tell if one pair of pcs is related under the same index number as another pair.[70] In other words, the student is asked to determine if the two pairs are both cycles of the same T_nI operator. Preparatory exercises could include distinguishing odd and even intervals and ics since index

[70]One could ask if the two pcs have the same axis of pitch inversion when the pcs are also related by T_nI in pitch. If not, then only the index number is used.

numbers that are odd involve odd intervals between pcs related by T_nI (the cycles) and even index numbers involve even intervals. Singing exercises such as those given in Example 7 provide some needed practice.

Example 7. T_nI cycle singing exercises

T_nI cycle singing exercise (n = odd = 9)

Divide into two groups; sing in any octave.

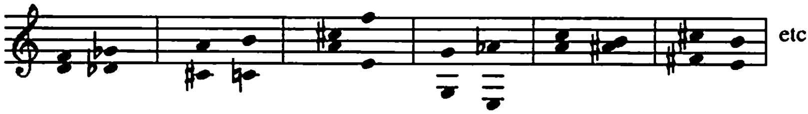
T_nI cycle singing exercise (n = even = 6)

Divide into two groups; sing in any octave.

Example 8a provides two sets of examples: (8a1) if the pairs share the same index number in pc they also share the same pitch index number; (8a2) the two pairs of pcs are not related by pitch inversion. Ultimately, all of the examples in 8a have to do with identifying the content of the cycles of the twelve T_nI operations. The student can also practice producing and hearing pitch realizations of sets of pc pairs related by the same index of inversion. A last stage involves identifying the index number relating two pairs of pcsegs presented successively or overlapped.

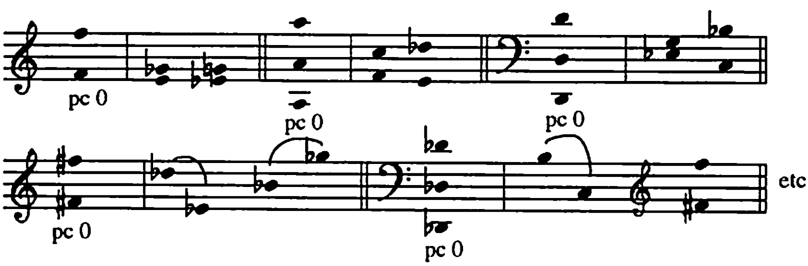
Example 8. Exercises in identification of index numbers

8a1. Do the following pairs of intervals share the same index number, or not? (Each example played 3 times.)



8a2. Do the following pairs of intervals share the same index number, or not? (Each example played 3 times.)

8b. Identify the index number of the pairs of intervals. (Each example played 3 times.)



Answers:

8a1: Y, N, N, N, Y, N

8a2: Y, Y, N, Y

8b: 0, e, 6, 4, 9

A final degree of finesse: to identify the transformation relating the pitch realizations of two pcsegs (ordered pcsets) without the benefit of contour relations produced by coordinating pitch and pitch-class transposition or inversions. Example 8b provides examples for identifying index numbers ranked by relative difficulty. These examples include an octave marked pc 0, which is played before the interval pairs to establish a basis for assessing the label for the index number.

Stage 4: Identifying some of the larger set-classes.

There are 29 tetrachordal set-classes. If we differentiate these further into their T_n-type set-classes and fb-classes the number of sets to learn to hear becomes too huge to manage in totality, at least in the span of one course alone. Fortunately, some of these set-classes will be familiar or easily memorized. In addition, a few of these tetrachords are found with great frequency in atonal music. A list of selected tetrachordal set-classes is given below.[71]

Common atonal set-classes:

4-1[0123]
4-9[0167]
4-19[0148]

All-interval tetrachords:

4-15[0146]
4-29[0137]

"Tonal" set-classes:

4-20[0158]	major-seventh-chord
4-25[0268]	french augmented-sixth-chord
4-26[0358]	minor-seventh-chord
4-27[0258]	half-diminished- or dominant-seventh-chord.
4-28[0369]	diminished-seventh-chord.

Combinatorial tetrachordal set-classes:

4-1[0123]	(also above)
4-6[0127]	
4-9[0167]	(also above)
4-10[0235]	
4-13[0136]	
4-23[0257]	
4-28[0369]	(also above)

[71]While Friedmann expects the student to be able eventually to identify all 29 tetrachordal set-classes, he drills a similar selection of these readily identifiable tetrachords in his Exercise 5.6 (pp. 80-81).

The all-interval tetrachords are Z-related, but their trichordal content (under abstract inclusion) differentiates them.[72] As we will see, they are easily constructed from the union of any ic 3 and ic 6 that do not intersect.

The pcsets members of the tetrachordal combinatorial set-classes can generate the aggregate in threes, mutually related by T_4 and T_8—hence their identification with combinatoriality. These seven set-classes also share other properties: (1) each of their members is invariant under T_nI except those of 4-13[0136]; (2) all of their pcset members omit ic 4. Once again, I stress the drill necessary to make tetrachordal identification direct and unpremeditated. As with trichords, exercises can be constructed based on contrasting tetrachordal set-classes to help secure immediate recognition.

In addition to the tetrachords, students are ready to learn to distinguish the six all-combinatorial hexachordal set-classes given earlier [see Martino's labels A through F] from each other and from the other 44 hexachordal set-classes. One might add nine more set-classes to the mix. These are given letters or nicknames for identification.[73] They are also accompanied by descriptions that help one to construct them without recourse to normal form or interval vector construction.

P	6-30[013679]	(The Petrushka chord and its inversion; two major or minor chords at the tritone.)
Q	6-14[013458]	(Any non-intersecting union of a member of 3-6[024] and a member of 3-12[048].)
S_1	6-44[012569]	(The Schoenberg "signature set;" two members of 3-3[014] related by T_5 or T_7.)

[72]Set-class 4-15[0146] abstractly includes 3-3[014] while 4-29[0137] abstractly includes 3-11[037].

[73]The names Q, S_1 and S_2 were inspired by Andrew Mead. He informally refers to S_1 and S_2 as the "twisted-sisters."

S_2	6-19[013478]	(The abstract complement of the Schoenberg signature; two members of 3-11[037] related by T_1 or T_e.)
m	5-12[01356]	("The Maverick sonority;" first five notes of the Locrian mode.)
d	7-35[013568t]	(The diatonic set; any of the "church modes.")
t	8-9[01236789]	(Messiaen's mode IV.)
O	8-28[0134679t]	(Octatonic scale; Messiaen's mode II; the abstract complement of 4-28[0369], the diminished-seventh-chord.)
N	9-12[01245689t]	(Nonotonic scale; Messiaen's mode III; the abstract complement of 3-12[048], the augmented chord.)

The P and Q hexachords are often found in aggregate formations (partitions) with the one or more of the six all-combinatorial hexachords.[74] S_1 and S_2 are Z-related. S_1 is the Schoenberg "signature set," i.e., the letters of Schoenberg's last name: S → Es → German for E-flat, C, H → German for B, B → German for B-flat, E, and G = {30et47} ∈ S_1. S_2 is the abstract complement (and M-correspondent[75]) of S_1.[76] With regard to set-class m, Howard Hanson referred to it as the "maverick sonority" because it is the only pentachordal set-class that is not abstractly included in its complement. m is also Z-related to 5-36[01247], but the two set-classes have pcsets with different degrees of symmetry. d is the diatonic scale and takes a lower

[74]See Morris and Alegant, "The Even Partitions in Twelve-Tone Music," *Music Theory Spectrum* 10 (1988): 74-101.

[75]That is, S_1's members are related to the members of S_2 by the M_5 or M_7 (multiplicative) operations.

[76]Schoenberg used S_1 and S_2 often in his early and middle periods. S_2 occurs at the opening of the second part of Stravinsky's *Rite of Spring*. An E-flat minor and C#-minor chord alternate over a low D minor chord. The combination of either of the oscillating chords and the D-minor chord comprises a S_2 sonority

case letter, not because the context is atonal music, but because capital D has been already reserved for an all-combinatorial hexachord. The name t is given to 8-9[01236789] since it contains pcsets rich in tritones. O and N, the octatonic and nonotonic scales, are found ubiquitously in the music of the late nineteenth and early twentieth century. Both were used extensively by Skryabin.[77]

Stage 5: Identifying the Hexachordal Set-classes

By now the student will be able to recognize at least ten hexachordal set-classes: A, B, C, D, E, F, P, Q, S_1, and S_2. Up to now I have not explicitly suggested any general method for memorizing set-classes; however, such methods easily fall out of the comments I have made about various set-classes in earlier stages, especially stage 4. The underlying strategies for learning set-classes are two—by construction from smaller set-classes that uniquely define the members of a given set-class and by grouping and associating set-classes and their members in families.[78]

1. Set-class construction: I use just two methods of definitively producing larger cardinalities of set-classes from a core of smaller ones: Transpositional Combination and Complement Union.

Transpositional Combination[79] (abbreviated *TC*) has been extensively studied by Richard Cohn.[80] In its simplest form, two

[77]Friedmann's list of important larger sets includes the all-combinatorial hexachords, S_1, S_2 and the unique abstract supersets or subsets of the octatonic, E, and F set-classes.

[78]These strategies are based on the two most fundamental aspects of human cognition: "chunking"(hierarchy) and "association"(similarity).

[79]Transpositional combination was first defined in Hanson *Harmonic Materials* where it is called "projection." "Multiplication" is Boulez's term for what is essentially the same thing; see Pierre Boulez, *Boulez on Music Today*, trans. Susan Bradshaw and Richard Rodney Bennett (London: Faber and Faber, 1971).

[80]See Cohn, "Inversional Symmetry and Transpositional Combination in Bartók," *Music Theory Spectrum* 10 (1988): 19-42 and "Transpositional

transpositions of a smaller set construct a larger set.[81] The two smaller sets are related by the interval n so we can say the larger set is the transpositional combination of the smaller set at interval n.[82] The above comments about the set-class S_2 (6-19[013478]) employs TC: "two members of 3-11[037] related by T_1 or T_e". That is, if we take a major or minor chord and combine it with a copy of itself transposed by the interval 1 or e, we produce a member of S_2. Since we are concerned only with the set-class membership of the combination of the smaller sets, the smaller of the two intervals can stand in for both. For example, any member of S_2 is the TC of any member of 3-11 at interval 1.[83] We write this relation in a kind of shorthand as:

$$S_2 = \text{3-11[037]@1}$$

To my knowledge, the *Complement Union Property (CUP)* was first defined and discussed in my article "Pitch-Class Complementation and its Generalizations."[84] A set-class ZZ has CUP if and only if any of its pcset members can be generated by the union of any member X of a set-class XX and any member Y of another set-class YY providing X and Y do not intersect. We say set-class ZZ is generated by set-classes XX and YY via CUP. The set-class Q (6-14[013458]) has members with this property. If one locates a 3-12[048] (augmented chord) in a member of Q, the remaining trichord is always a member of 3-6[024] as stated

Combination in Twentieth-Century Music" (PhD diss., Eastman School of Music, University of Rochester, 1986).

[81]Friedmann emphasizes transpositional combination (called by him multiplication) in his presentation of melodies for all T_nI-invariant set-classes of four or more elements. See his Exercise 5.7 (p. 82) and his discussion of S_1 and S_2 on p.110.

[82]If the smaller set is X and the larger is Y, then TC can be defined as a function $Y = TC(X,n) = X \cap T_nX$.

[83]Let X and T_nX define TC. Then $T_{-n}X$ and $T_{-n}T_nX$ (= X) also produce TC defining $T_{-n}Y$, which is in the same set-class as Y. Thus the n in TC can always be substituted by -n when constructing members of the set-class of Y via TC.

[84]Morris, "Pitch-Class Complementation and its Generalizations," *Journal of Music Theory* 34/2 (1990): 175-246.

above. The "if and only if" part of the definition of CUP means that if we select any member of 3-12[048] and any member of 3-6[024] the union (provided the two members share no pcs) is always a member of Q. We notate this relation as follows:

Q = 3-6[024] & 3-12[048]

Sometimes CUP connects two Z-related set-classes as in the following. Note the identifying "Z:" prefix.

Z: S_2 = 3-8[026] & 3-12[048]

implies

Z: S_1 = 3-8[026] & 3-12[048]

and vice versa. Here non-intersecting members of 3-8[026] and 3-12[048] in union will form hexachords that are members of either S_1 or S_2.

We can also write:

Z: 4-15[0146] / 4-29[0137] = ic 3 & ic 6 (or ic 6 & ic 3).

In sum, one can easily recall or identify set-classes by their TC and CUP relations. For instance: a member of set-class Q can be constructed by combining any non-intersecting augmented chord and a member of 3-6[024] (like {048} and {79e}); a member of P can be made by taking two members of the 3-11[037] set-classes a tritone apart (like {259} and {38e}); an all-interval tetrachord can be generated by the non-intersecting union of any tritone and any minor third (like {17} and {36}).

When attempting to determine the SC of a six-pc pcset using CUP, one should look in the pcset for the T_n-invariant pcset first. For instance, in the case of 6-17[012478], CUP is defined as 3-12 & 3-5. If one looks for the 3-5 first, one will readily find one of the four 3-5s that are included in each member of 6-17. The problem is that only one of these 3-5s has 3-12 as its CUP partner; thus, it is likely that the 3-5 found will not be the one that produces CUP with 3-12 and one might assume that the pcset is not a member of 6-17. On the other hand, if one looks for 3-12 first, then there is no ambiguity. Each 6-17 member has only one 3-12 whose remaining three pcs are of 3-5. So by looking for the transpositional invariant member of CUP first such uncertainties are removed.

The Appendix includes TC and CUP relations for all hexachordal set-classes.

2. Set-class Association: This can take various forms. We can remember a set-class because one or more of its members is a "celebrated" music object. We can memorize set-class P by calling up the Petrushka chord (a C major triad "against" an F-sharp major triad). F is the set of whole-tone scales. Or the first six pcs of the row of Berg's Violin Concerto form a member of set-class 6-24[013468]. This form of association is known as "associated repertoire."

Sorting set-classes in groups can also stimulate memorization.[85] Above, I assumed sets that were important tonal entities would be easily memorized as atonal objects. Refining and generalizing this notion, set-classes can be related to each other by their abstract inclusion in a larger set-class. For instance, while none of the all-combinatorial hexachords is abstractly included in the octatonic set-class, P is so included. Other categories beside octatonic include "diatonic" (those set-classes abstractly included in the major scale (7-35[013568t]) "chromatic" (included in the A set-class or a larger set-class such

[85]Friedmann uses this strategy to group trichords and tetrachordal set-classes into categories. For tetrachords he has three categories. (1) Three "families": family 1 contains pcsets that include ic 1; family 2 has pcsets that include ic 2 but not ic 1; family 3 has pcsets without ic 1 or ic 2. This separates set-classes roughly into chromatic, whole-tone, and diatonic groups. Such categories are generalized by Eriksson; see "The Max Point." (2) Five groups: pcsets that are abstractly included in the diatonic scale, and/or octatonic, or whole-tone set-class, or abstractly include 3-1[012] and/or 3-12[048]. (3) By whole-tone-scale content: pcsets are members of the same group if they have n pcs in one member of set-class F, and m pcs in the other member of F, where m + n total the cardinality of the pcset. Friedmann states he has borrowed this idea from Andrew Mead, "Pedagogically Speaking: A Practical Method for Dealing with Unordered Pitch-Class Collections," *In Theory Only* 7/5-6(1984): 54-66. It can be developed further to register the number of pcs of a pcset in each cycle of the set of cycles of each T_n operator.

as 7-1[0123456] or 8-1[01234567]), "whole-tone" (included in F), and so forth. Other kinds of set-class families have pcsets that share special properties. The prime example is the set of all-combinatorial hexachords.[86]

Another special property is the all-x-chord attribute, shared by the following set-classes:

> 4-15[0146] and 4-29[0137] each abstractly contains every set-class of cardinality 2.
>
> 6-17[012478] abstractly contains every set-class of cardinality 3.
>
> 8-15[01234689] and 8-29[01235679] each abstractly contains every set-class of cardinality 4.

Still another family is composed of Z-related hexachord pairs. This family is called PZ3-12 in the Appendix. Let X be one member of each pair such that X is composed of a member of 3-12[048] and another non-intersecting trichord Y; the Z-related partner, the complement of X, is composed of the union of T_4Y and T_8Y. An example: X is a member of set-class 6-39[023458], X′ is a member of 6-10[013457], the abstract complement and Z-correspondant of 6-39[023458], and Y is a member of 3-2[013].

> {023} = Y, {467} = T_4Y, {8te} = T_8Y, {159} is a member of 3-12[048].
>
> {023} $\cup$ {159} $\in$ 6-39[023458] and {467} $\cup$ {6te} $\in$ 6-10[013457].

These relations can be expressed in terms of TC and CUP:

> 6-10[012457] = 3-2[013] @ 4.
>
> 6-39[023458] = 3-2[013] & 3-12[048].

The Appendix contains "The Hexachordal Set-Class Table," a comprehensive list of the 50 hexachordal set-classes related by some of the methods given above and others. Study of the table

[86] A set-class can be easily remembered if it has striking and/or unique properties. For instance, Q is the only hexachordal set-class whose members can map onto their complements under transposition alone. P's members are the only hexachords that can map onto their complements by inversion alone and are invariant under T_0 and T_6. These facts will be well known to a twelve-tone composer.

makes it possible for one to teach and/or learn to identify any arbitrary set of six distinct pcs as to set-class, without the use of normal-forms, interval-class vectors or table look-ups. Every hexachordal set-class can be generated by TC or CUP. The Notes on the Appendix describe the table in detail.

Of course, one need not memorize the entire table at once; it can be used to keep a particular group of hexachords in mind for a limited period of time, during analysis or composition.[87] In fact, it may be a good idea to leave concerted study of the last columns of the table until one has grown accustomed to thinking of the hexachords in terms of TC and CUP. The Property Families group the hexachords by their CUP relations. It is quite remarkable that CUP can connect Z-related hexachords in various families.[88] And despite its elegance, CUP relations are not fully understood yet.[89] The "T inv", "I inv", "T comp map" and, "I comp map" columns are equivalent to abridged *invariance-vectors* (omitting invariances and complement mapping under the M and MI (M_5 and M_7) multiplicative operators).[90]

Final Remarks

It is an unfortunate fact that in many modern curricula the development of musicianship has been kept more or less separate from the study of analysis and composition. This can be remedied by the inclusion of more music literature and analysis in musicianship classes. Without doubt, there is a need for more anthologies of twentieth-century music especially for use in classes studying atonal music.[91] The transcriptions at the end of the Friedmann text are a

[87]One can tackle the hexachords in groups, beginning with any of the inclusion or property families listed in the notes to the appendix.

[88]In contrast, when grouping hexachords according to inclusion families, the members of a Z-pair are not both obliged to be in the same family, even though their interval-class content is identical.

[89]See Morris, "Pitch-Class Complementation."

[90]Consult set-class tables in Morris, *Class Notes*, or *Composition* for complete invariance vectors.

[91]Among anthologies of twentieth-century music are: Thomas Delio and Stuart Saunders Smith, *Twentieth-Century Music Scores* (Englewood

good start. Perhaps a database will be established and keyed for pedagogical purposes. But beyond literature, students also need to see good and diverse examples of high-level atonal analysis.[92] Rahn and Straus furnish good introductions to analysis[93] while more difficult but quite satisfying work is exemplified by recent articles on Schoenberg's music by Andrew Mead and Allen Forte.[94]

I shall end with a quotation of John Cage: "Composing's one thing, performing's another, listening's a third. What can they have to do with one another?"[95] While Cage often answered his query in the negative, those of us who have chosen to live some of our musical lives in atonal music cannot agree. Sharing our responses to music puts us in touch with the question, but teaching and learning help answer it.[96]

Cliffs, NJ:Prentice Hall, 1989); Bryan R. Simms, *Music of The Twentieth Century: An Anthology* (New York: Schirmer Books, 1986); and Mary H. Wennerstrom, *Anthology of Twentieth-Century Music, 2nd ed.* (Englewood Cliffs, NJ:Prentice Hall, 1988). Such anthologies and those that include twentieth-century music in a larger time frame, as they should, produce samples from the vast diversity of contemporary music. Whereas at least half of this music can be addressed in part by atonal music theory tools, most of the pieces that are most amenable to atonal analysis are not found in these collections.

[92]A bibliography of theoretical and analytical literature was published in volume 11 of *Music Theory Spectrum*. See Martha Hyde, "Twentieth-Century Analysis during the Past Decade: Achievements and New Directions," *Music Theory Spectrum* 11 (1989): 35-39 and Andrew Mead, "The State of Research in Twelve-Tone and Atonal Music Theory," *Music Theory Spectrum* 11 (1989): 40-48.

[93]See Rahn, *Basic Atonal Theory* (4-18, 59-73); Straus, *Introduction* (16-25, 47-58, 79-89, 110-117, 136-146, 169-179.)

[94]Andrew Mead, "Large-Scale Strategy in Arnold Schoenberg's Twelve-Tone Music," *Perspectives of New Music* 24/1 (1985): 120-157 and Allen Forte, "Concepts of Linearity in Schoenberg's Atonal Music: A Study of the Opus 15 Cycle," *Journal of Music Theory* 36/2(1992): 285-382.

[95]John Cage, *Silence*, 15.

[96]I would like to thank Dora Hanninen, Ellen Koskoff, Michael Friedmann, Robert Gauldin, and Elizabeth Marvin for reading various drafts of this article and offering suggestions for its improvement. Any mistakes are mine, of course.

Appendix

The Hexachordal Set-Class Table

Name	AKA (short name)	ic-vector	Z	TC (Transpositional Combination)	CUP (Complement Union Property)	Inclusion Family	Property Family	T_n inv	T_nI inv	T_n comp map	T_nI comp map
6-1[012345]	A, chromatic	[543210]		3-1[012] @ 3			All-comb	1	1	1	1
6-2[012346]		[443211]		3-3[014] @ 2	6 & 4-1[0123]		P2-6	1	0	0	1
6-3[012356]		[433221]	6-36[012347]	3-7[025] @ 1	Z: 6 & 4-2[0124]	m	PZZ2-6	1	0	0	0
6-4[012456]		[432321]	6-37[012348]	3-1[012] @ 4	Z: 6 & 4-3[0134]	N	PZZ2-6 PZ3-12	1	1	0	0
6-5[012367]		[422232]		3-8[026] @ 1		t		1	0	0	1
6-6[012567]		[421242]	6-38[012378]	3-1[102] @ 5	Z: 3 & 4-9[0167]	t	MZ PZZ4-9	1	1	0	0
6-7[012678]	D	[420243]		3-1[012] @ 6 3-4[015] @ 6 3-9[027] @ 6		t	All-comb	2	2	2	2
6-8[023457]	B	[343230]		3-6[024] @ 3			All-comb	1	1	1	1

6-9[012357]		[342231]		3-4[015] @ 2	6 & 4-10[0235]		P2-6	1	0	0	1
6-10[013457]		[333321]	6-39[023458]	3-2[013] @ 4	Z: 6 &4-4[0125]	N	PZ3-12	1	0	0	0
							PZZ2-6				
6-11[012457]		[333231]	6-40[012358]		Z: 6 & 4-11[0135]		MZ	1	0	0	0
							PZZ2-6				
6-12[012467]		[332232]	6-41[012368]		2 & 4-9[0167]	m	PZ4-9	1	0	0	0
6-13[013467]		[324222]	6-42[012369]	3-10[036] @ 1		O	PZ4-28	1	1	0	0
6-14[013458]	Q	[323430]		3-4[015] @ 3	3-6[024] & 3-12[048]	N		1	0	1	0
6-15[012458]		[323421]			6 & 4-7[0145]	N	P2-6	1	0	0	1
6-16[014568]		[322431]			6 & 4-17[0347]	N	P2-6	1	0	0	1
6-17[012478]	All-trichord	[322332]	6-43[012568]		3-5[016] & 3-12[048]		PZ3-12	1	0	0	0
					4 & 4-9[0167]		PZ4-9				
6-18[012578]		[322242]		3-8[026] @ 5		t		1	0	0	1
6-19[013478]	S_2	[313431]	6-44[012569]	3-11[037] @ 1	Z: 6 & 4-19[0148]	N	MZ	1	0	0	0
					Z: 3-8[026] & 3-12[048]		PZZ2-6				
6-20[014589]	E, magic,	[303630]		3-12[048] @ 1		N	All-comb	3	3	3	3
	hexatonic			3-12[048] @ 3							
				3-12[048] @ 5							

6-21[023468]	[242412]			1 & 4-25[0268]	N	P4-25	1	0	0	1
6-22[012468]	[421422]			3 & 4-25[0268]	N	P4-25	1	0	0	1
6-23[023568]	[234222]	6-45[023469]	3-10[036] @ 2		O	PZ4-28	1	1	0	0
6-24[013468]	[233331]	6-46[012469]		Z: 6 & 4-14[0237]		PZ3-12	1	0	0	0
				3-7[025] & 3-12[048]						
6-25[013568]	[233241]	6-47[012479]	3-2[013] @ 5	Z: 6 & 4-22[0247]	d m	PZZ2-6	1	0	0	0
6-26[013578]	[232341]	6-48[012579]	3-9[027] @ 4	Z: 6 & 4-26[0358]	d N	PZ3-12	1	1	0	0
						PZZ2-6				
6-27[013469]	[225222]		3-5[016] @ 3	3 & 4-28[0369]	O		1	0	0	1
			3-8[026] @ 3							
6-28[013569]	[224322]	6-49[013479]		4 & 4-28[0369]	m	PZ3-12	1	1	0	0
				3-10[036] & 3-12[048]		PZ4-28				
6-29[023679]	[224232]	6-50[014679]		5 & 4-28[0369]	t	PZ4-28	1	1	0	0
6-30[013679] P, Petrushka	[224223]		3-2[013] @ 6	6 & 4-28[0369]	O t		2	0	0	2
			3-3[014] @ 6							
			3-7[025] @ 6							
			3-11[037] @ 6							
6-31[014579]	[223431]			6 & 4-20[0158]	N	P2-6	1	0	0	1

6-32[024579]	C, Guidonian	[143250]		3-6[024] @ 5		d	All-comb	1	1	1	1
				3-9[027] @ 3							
6-33[023579]		[143241]		3-11[037] @ 2	6 & 4-23[0257]	d	P2-6	1	0	0	1
6-34[013579]		[142422]			5 & 4-25[0268]	N	P4-25	1	0	0	1
6-35[02468t]	F, whole-tone	[060603]		3-12[048] @ 2		N	All-comb	6	6	6	6
				3-12[048] @ 6							
6-36[012347]		[433221]	6-3[012356]		Z: 6 & 4-2[0124]		PZZ2-6	1	0	0	0
6-37[012348]		[432321]	6-4[012456]		Z: 6 & 4-3[0134]		PZ3-12	1	1	0	0
					3-1[012] & 3-12[048]		PZZ2-6				
6-38[012378]		[421242]	6-6[012567]	3-9[027] @ 1	Z 3 & 4-9[0167]	t	MZ	1	1	0	0
							PZZ4-9				
6-39[023458]		[333321]	6-10[013457]		Z: 6 & 4-4[0125]		PZ3-12	1	0	0	0
					3-2[013] & 3-12[048]		PZZ2-6				
6-40[012358]		[333231]	6-11[012457]		Z: 6 & 4-11[0135]		MZ	1	0	0	0
							PZZ2-6				
6-41[012368]		[332232]	6-12[012467]	3-5[016] @ 2		t	PZ4-9	1	0	0	0
6-42[012369]		[324222]	6-13[013467]		1 & 4-28[0369]	t	PZ4-28	1	1	0	0

6-43[012568]	[322332]	6-17[012478]	3-5[016] @ 4		N t	PZ3-12 PZ4-9	1	0	0	0
6-44[012569] S_1, Schoenberg signature	[313431]	6-19[013478]	3-3[014] @ 5	Z: 6 & 4-19[0148] Z: 3-8[026] & 3-12[048]	N	MZ PZZ2-6	1	0	0	0
6-45[023469]	[234222]	6-23[023568]		2 & 4-28[0369]		PZ4-28	1	1	0	0
6-46[012469]	[233331]	6-24[013468]	3-7[025] @ 4	Z: 6 & 4-14[0237]	N	PZ3-12	1	0	0	0
6-47[012479]	[233241]	6-25[013568]		Z: 6 & 4-22[0247]		PZZ2-6	1	0	0	0
6-48[012579]	[232341]	6-26[013578]		Z: 6 & 4-26[0358] 3-9[027] & 3-12[048]		PZ3-12 PZZ2-6	1	1	0	0
6-49[013479]	[224322]	6-28[013569]	3-10[036] @ 4		N O	PZ4-28 PZ3-12	1	1	0	0
6-50[014679]	[224232]	6-29[023679]	3-10[036] @ 5		O	PZ4-28	1	1	0	0

Notes on the Hexachordal Set-Class Table

Name
(column 1)

The set-class name (first Forte, then Rahn names).

AKA
(column 2)

The letter or other name for the set-class, if any.

ic-vector
(column 3)

The interval-class vector for the members of the set-class.

Z
(column 4)

The name of the Z-related set-class, if any.

TC
(column 5)

Any Transpositional Combinations (TC) that generate the set-class. (Single numbers stand for ics; for instance, "1" stands for ic 1.) See main body of paper for more.

CUP
(column 6)

Indicates the set-class's Complement Union Properties (CUP), if any. (As above, single numbers stand for ics.) See main body of paper for more.

Inclusion Families
(column 7)

Indicates the inclusion families of the set-class, if any.

d Family

Includes Set-classes that are abstractly included in set-class d (diatonic family 7-35[013568t]).

d family members:

6-25[013568]
6-26[013578]
6-32[024579] (C)
6-33[023579]

t Family

Includes Set-classes that are abstractly included in set-class t (tritone family 8-9[01236789]).

t family members:

6-5[012367]
6-6[012567] / 6-38[012378]
6-7[012678] (D)
6-18[012578]
6-29[023679]
6-30[013679] (P)
6-41[012368]
6-42[012369]
6-43[012568]

O Family

Includes Set-classes that are abstractly included in set-class O (Octatonic family 8-28[0134679t]).

O family members:

6-13[013467]
6-23[023568]
6-27[013469]
6-30[013679] (P)
6-49[013479]
6-50[014679]

N Family

Includes Set-classes that are abstractly included in set-class N (Nonotonic family 9-12[01245689t]).

N family members:

6-4[012456]
6-10[013457]
6-14[013458] (Q)
6-15[012458]
6-16[014568]
6-19[013478] / 6-44[012569] (S_2/S_1)
6-20[014589] (E)
6-21[023468]
6-22[012468]
6-26[013578]
6-31[014579]
6-34[013579]
6-35[02468t] (F)
6-43[012568]
6-46[012469]
6-49[013479]

m Family

Includes Set-classes that abstractly include set-class m (maverick family 5-12[01356]).

m family members:

6-3[012356]
6-12[012467]
6-25[013568]
6-28[013569]

Property Families
(column 8)

Indicates the property families of the set-class, if any.

All-Comb Property Family

The all combinatorial hexachordal set-classes. Each member has invariance under T_nI, T_n complement mapping, T_nI complement mapping (for at least one n).

All-Comb family members:

Hexachords (AKA column:) A, B, C, D, E, and F.

MZ Property Family

Z-related hexachordal set-classes also abstractly related by M/MI (M_5/M_7) multiplicative operations.

MZ family members:

6-6[012567] / 6-38[012378]
6-11[012457] / 6-40[012358]
6-19[013478] / 6-44[012569] (S_2 / S_1)

PZ3-12 Property Family

Z-related Hexachordal set-classes also related by TC or CUP. Let A be any trichordal pcset without ic4; A is a member of the set-class AA.

AA @ 4 specifies half the set-classes in this family. The other half, Z-related to the members of the first half, are specified by AA & 3-12.

PZ3-12 family members (first TC member, slash, then CUP member):

6-4[012456] / 6-37[012348]	AA = 3-1
6-10[013457] / 6-39[023458]	AA = 3-2
6-26[013578] / 6-48[012579]	AA = 3-9
6-43[012568] / 6-17[012478]	AA = 3-5
6-46[012469] / 6-24[013468]	AA = 3-7
6-49[013479] / 6-28[013569]	AA = 3-10

PZ4-28 Property Family

Z-related Hexachordal set-classes also related by TC or CUP. Let A be any ic except ic3 or ic6;

A @ 3-10[036] specifies half the set-classes in this family. The other half, Z-related to the members of the first half, are specified by A & 4-28[0369].

PZ4-28 family members (first TC member, slash, then CUP member):

6-13[013467] / 6-42[012369]	A = ic 1
6-23[023568] / 6-45[023469]	A = ic 2
6-49[013479] / 6-28[013569]	A = ic 4
6-50[014679] / 6-29[023679]	A = ic 5

P2-6 and **PZZ2-6** Property Families

P2-6 and PZZ2-6 have members defined by CUP: A & 6, where A is a tetrachordal set-class whose members omit ic 6.

In P2-6, A's pcsets have T_nI invariance under an even index number (n=even).

P2-6 family members:

6-2[012346]	A = 4-1
6-9[012357]	A = 4-10
6-15[012458]	A = 4-7
6-16[014568]	A = 4-17
6-31[014579]	A = 4-20
6-33[023579]	A = 4-23

In PZZ2-6, A's pcsets do not have T_nI invariance under an even index number (n=even). A & 6 produces the members of (both) Z-related set-classes

PZZ2-6 family members

6-3[012356] / 6-36[012347]	A = 4-2
6-4[012456] / 6-37[012348]	A = 4-3
6-10[013457] / 6-39[023458]	A = 4-4
6-11[012457] / 6-40[012358]	A = 4-11
6-19[013478] / 6-44[012569] (S_2 / S_1)	A = 4-19
6-25[013568] / 6-47[012479]	A = 4-22
6-26[013578] / 6-48[012579]	A = 4-26

P4-25 Property Family

P4-25 have members defined by CUP: A & 4-25, where A is ic 1, 3, or 5.

P4-25 Members:

6-21[023468]	A = ic 1
6-22[012468]	A = ic 3
6-34[013579]	A = ic 5

PZ4-9 and **PZZ4-9** Property Families

PZ4-9 and PZZ4-9 have members defined by CUP: A & 4-9, where A is ic 2, 3, or 4.

PZ4-9 members are Z-related hexachords, but the CUP concerns only one of the two Z-correspondants. The other Z-correspondent has TC: 3-5 @ A.

PZ4-9 family members (First TC member, slash, then CUP member):

6-41[012368] / 6-12[012467]	A = ic 2
6-43[012568] / 6-17[012478]	A = ic 4

PZZ4-9 has only one Z-pair of members. A is ic 3 so CUP = 4-9 @ 3 which generates both members of the Z-pair.

PZZ4-9 family members:

6-6[012567] / 6-38[012378]	A = ic 3

NB: The is a logic to property family names. The P stands for Property; the last part of the name is the constant set-class. The name has: no "Z", if Z-relations are not involved; 1 "Z", if one member of a Z-relation pair is generated by CUP and the other by TC (with the same variable set-class (always named A above)); 2 "Z"s ("ZZ"), if both members of the Z-relation pairs are generated by the same CUP.

T inv I inv T comp map I comp map
(columns 9, 10 ,11 ,12)

These last columns spell out the invariances of the members of each set-class.

T inv (column 9): the number of distinct T_n invariances for each member of the set-class. (All pcsets have identity (T_0) invariance, so a 1 entry stands for identity invariance.)

I inv (column 10): the number of distinct T_nI invariances for each member of the set-class. (A blank entry stands for zero T_nI invariances.)

T comp map (column 11): the number of T_n operations that map each member of the set-class to its complement.

I comp map (column 12): the number of T_nI operations that map each member of the set-class to its complement.

Example: 6-1[012345] has in columns 9 to 12: 1 1 1 1. This means each of its members has invariance under 1 case each of T_0 (identity), and T_nI invariance and 1 case each of T_n and T_nI of complement mapping.

Another example: 6-30[013679] has in columns 9 to 2: 2 0 0 2. This means each of its members has invariance under 2 cases of T_n (one of which is identity T_n), and 2 cases of T_nI complement mapping.

The reader can use these last columns to determine a set-classes's hexachordal combinatorial properties in twelve-tone theory:

If column 9 is 2 or more, the hexachord can be used to generate non-trivial R-type combinatoriality.

If column 10 is 1 or more, the hexachord can be used to generate RI-type combinatoriality.

If column 11 is 1 or more, the hexachord can be used to generate T-type combinatoriality.

If column 12 is 1 or more, the hexachord can be used to generate I-type combinatoriality.

(The all-combinatorial hexachords have 1 or more in all four columns.)